Math 2411B **Quiz One** **September 12, 2000**

You may use any books, notes, tables, calculators, or computers you wish. Be sure and write so that someone other than yourself can understand your exposition.

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1. Let C be the curve described by

$$\mathbf{r}(t) = (t^3 + 7)\mathbf{i} + 2t\mathbf{j} + (t^2 - 1)\mathbf{k}.$$

Find *all* points on C at which the tangent line is parallel to the line

$$l(t) = (5 + 6t)\mathbf{i} + (9 + t)\mathbf{j} + (1 - 2t)\mathbf{k},$$

or explain carefully why there are no such points.

A straight line in the direction of a vector $\mathbf{b}$ is described by a function of the form $l(t) = \mathbf{a} + t\mathbf{b}$. Thus a vector $\mathbf{b}$ in the direction of the given line is $\mathbf{b} = 6\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
Now, a vector tangent to the curve C at the point $\mathbf{r}(t)$ is simply the derivative

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k}.$$

For $\mathbf{b}$ and $\mathbf{r}'(t)$ to have the same direction, it must be true that $\mathbf{r}'(t) = \alpha\mathbf{b}$ for some scalar α. In other words,

$$3t^2 = 6\alpha$$
$$2 = \alpha$$
$$2t = -2\alpha$$

There is exactly one solution to this system of three equation, $t = -2$. Thus there is exactly one point, $\mathbf{r}(-2) = -\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, on C at which the tangent is parallel to the line described by $l(t)$.

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2. Find the curvature and torsion of the curve described by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}.$$

The curvature $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$ and $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$. Also, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, and $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$, where τ is the torsion. We need to compute all these. Here we go.

$$\mathbf{r}'(t) = -\sin t\mathbf{i} - \sin t\mathbf{j} + \sqrt{2}\cos t\mathbf{k}$$

Thus,

$$|r'(t)| = \sqrt{\sin^2 t + \sin^2 t + 2\cos^2 t} = \sqrt{2},$$

and so

$$\mathbf{T} = \frac{r'(t)}{|r'(t)|} = \frac{1}{\sqrt{2}}\left(-\sin t\mathbf{i} - \sin t\mathbf{j} + \sqrt{2}\cos t\mathbf{k}\right).$$

Continuing,

$$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{2}}\left(-\cos t\mathbf{i} - \cos t\mathbf{j} - \sqrt{2}\sin t\mathbf{k}\right).$$

Thus,

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} = \frac{d\mathbf{T}}{dt}\bigg/\frac{ds}{dt} = \frac{1}{\sqrt{2}}\frac{d\mathbf{T}}{dt}$$
$$= \frac{1}{2}\left(-\cos t\mathbf{i} - \cos t\mathbf{j} - \sqrt{2}\sin t\mathbf{k}\right)$$

So the curvature

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{2}.$$

Note that $\mathbf{N} = -\cos t\mathbf{i} - \cos t\mathbf{j} - \sqrt{2}\sin t\mathbf{k}$.

For the torsion, we need the binormal $\mathbf{B} = \mathbf{T} \times \mathbf{N}$:

$$\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & -\sin t & \sqrt{2}\cos t \\ -\cos t & -\cos t & -\sqrt{2}\sin t \end{vmatrix}$$
$$= \sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j}$$

The binormal is constant, and so its derivative is zero–hence the torsion $\tau = 0$.

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