Math 2411 **Quiz Five** **November 9, 2000**

You may use any books, notes, tables, calculators, or computers you wish. Write your answers so that someone other than yourself can understand your exposition. Please do not return this sheet.

1. A uniform piece of wire has the shape of the arc of the circle $x^2 + y^2 = 1$ in the first quadrant joining the points $(a, 0)$ and $(a\cos\alpha, a\sin\alpha)$, where α is between 0 and $\pi/2$. Find its center of mass.

The center of mass $(\tilde{x}, \tilde{y})$ is given by

$$\tilde{x} = \frac{\int_W x\,ds}{\int_W ds} \text{ and } \tilde{y} = \frac{\int_W y\,ds}{\int_W ds}.$$

(The constant density cancels top and bottom.)
A vector description of the wire W: $r(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le \alpha$. Thus, $r'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$, and so $|r'(t)| = a$. Now for the integrals:

$$\int_W x\,ds = \int_0^\alpha x|r'(t)|dt = \int_0^\alpha a^2\cos t\,dt = a^2\sin\alpha$$

$$\int_W y\,ds = \int_0^\alpha a^2\sin t\,dt = a^2(1-\cos\alpha)$$

$$\int_W ds = \int_0^\alpha a\,dt = a\alpha$$

And so,

$$\tilde{x} = \frac{\int_W x\,ds}{\int_W ds} = \frac{a^2\sin\alpha}{a\alpha} = a\frac{\sin\alpha}{\alpha}, \text{ and}$$

$$\tilde{y} = \frac{\int_W y\,ds}{\int_W ds} = \frac{a^2(1-\cos\alpha)}{a\alpha} = a\frac{(1-\cos\alpha)}{\alpha}$$

2. Let $\mathbf{F}(x,y) = (x^2y - 3x)\mathbf{i} + xy\mathbf{j}$.

a)Find the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

A vector description of C : $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 1$. And so $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$. Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^1 [(t^2t^2 - 3t)\mathbf{i} + tt^2\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j})dt$$

$$= \int_0^1 (t^4 - 3t + 2t^4)dt = -\frac{9}{10}$$

b) a)Find the integral $\int_D \mathbf{F} \cdot d\mathbf{r}$, where D is the curve $x = y^2$ from $(0,0)$ to $(1,1)$.

A vector description of D : $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$, and $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$. Thus

$$\int_D \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [(t^4t - 3t^2)\mathbf{i} + t^2t\mathbf{j}] \cdot (2t\mathbf{i} + \mathbf{j})dt$$

$$= \int_0^1 (2t^6 - 6t^3 + t^3)dt = -\frac{27}{28}.$$

c)Find a function g such that $\nabla g = \mathbf{F}$, or explain carefully why there is no such function.

There's no such function. The answers to Parts a) and b) show that the vector line integral from $(0,0)$ to $(1,1)$ is not path independent.