The Bead Game

We have a bead sliding down a wire joining two points in space, and want to find how long the trip will take. Suppose the shape of the wire is described by the vector function $\mathbf{R}(u)$, for $u_0 \le u \le u_1$. Thus the bead slides along the wire from the point $\mathbf{R}(u_0)$ to the point $\mathbf{R}(u_1)$. We assume the bead is initially at rest, and, of course, we must have $(\mathbf{R}(u_0) - \mathbf{R}(u_1)) \cdot \mathbf{k} > 0$ in order for the bead to slide "downhill."

Let g denote the gravitational acceleration. In falling from $\mathbf{R}(u_0)$ to $\mathbf{R}(u)$, a bead of mass m loses potential energy $mg(\mathbf{R}(u_0) - \mathbf{R}(u)) \cdot \mathbf{k}$, and gains kinetic energy $mv^2/2$, where v is the speed of the bead. These must be the same:

$$mg(\mathbf{R}(u_0) - \mathbf{R}(u)) \cdot \mathbf{k} = \frac{m}{2}\left|\frac{d\mathbf{R}}{dt}\right|^2.$$

(Needless to say, we assume no friction.). Now, $\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{R}}{du}\frac{du}{dt}$. Thus,

$$2g(\mathbf{R}(u_0) - \mathbf{R}(u)) \cdot \mathbf{k} = \mathbf{R}'(u) \cdot \mathbf{R}'(u)\left(\frac{du}{dt}\right)^2, \text{ and}$$

$$\left|\frac{du}{dt}\right| = \sqrt{\frac{2g(\mathbf{R}(u_0) - \mathbf{R}(u)) \cdot \mathbf{k}}{\mathbf{R}'(u) \cdot \mathbf{R}'(u)}}$$

If we assume that the parameter u increases with time, then $\frac{du}{dt} > 0$, and we have

$$\frac{dt}{du} = \sqrt{\frac{\mathbf{R}'(u) \cdot \mathbf{R}'(u)}{2g(\mathbf{R}(u_0) - \mathbf{R}(u)) \cdot \mathbf{k}}}.$$

Then the time T for the trip from $\mathbf{R}(u_0)$ to $\mathbf{R}(u_1)$ is simply

$$T = \int_0^T dt = \int_{u_0}^{u_1} \frac{dt}{du} du = \int_{u_0}^{u_1} \sqrt{\frac{\mathbf{R}'(u) \cdot \mathbf{R}'(u)}{2g(\mathbf{R}(u_0) - \mathbf{R}(u)) \cdot \mathbf{k}}}\, du.$$

Let's try this on an easy and obvious example first. To begin, suppose the bead starts at $(0,0,h)$ and ends at $(0,0,0)$, falling straight down. In this case, we have

$$\mathbf{R}(u) = (h-u)\mathbf{k},\ 0 \le u \le h.$$

Then,

$$
\begin{aligned}
T &= \int_0^h \sqrt{\frac{\mathbf{R}'(u)\cdot\mathbf{R}'(u)}{2g(\mathbf{R}(0)-\mathbf{R}(u))\cdot\mathbf{k}}}\,du \\
&= \int_0^h \sqrt{\frac{1}{2gu}}\,du = \frac{1}{\sqrt{2g}}\int_0^h \frac{1}{\sqrt{u}}du = \sqrt{\frac{2h}{g}},
\end{aligned}
$$

exactly the result we got in Mr. Crews's Fifth Grade Physics Class!

Next, let's look at the case in which the bead falls from the point $(a,0,h)$ to the point $(a,0,0)$ directly below by sliding along the helix described by

$$\mathbf{R}(u) = a\cos u\mathbf{i} + a\sin u\mathbf{j} + \frac{h}{2n\pi}u\mathbf{k},\ 0 \le u \le 2n\pi.$$

Note this is a helical path which the bead turns around n times on its way down from $(a,0,h)$ to $(a,0,0)$.

Let's begin:

$$\mathbf{R}'(u) = -a\sin u\mathbf{i} + a\cos u\mathbf{j} + \frac{h}{2n\pi}\mathbf{k},\ \text{and so}$$

$$\mathbf{R}'(u)\cdot\mathbf{R}'(u) = a^2 + \left(\frac{h}{2n\pi}\right)^2 = \frac{(2n\pi a)^2 + h^2}{(2n\pi)^2}$$

There is a slight compication in this case because the parameter *u decreases with time*. Thus $\frac{du}{dt} < 0$.

$$
\begin{aligned}
T &= -\int_{2n\pi}^{0} \sqrt{\frac{(2n\pi a)^2 + h^2}{(2n\pi)^2 2g(h - hu/2n\pi)}}\,du \\
&= \frac{1}{2n\pi}\sqrt{\frac{(2n\pi a)^2 + h^2}{2gh}}\int_0^{2n\pi} \frac{1}{\sqrt{1 - u/2n\pi}}du
\end{aligned}
$$

Now, let $\xi = u/2n\pi$. Then we have

$$
\begin{aligned}
T &= \sqrt{\frac{(2n\pi a)^2 + h^2}{2gh}}\int_0^1 \frac{1}{\sqrt{1-\xi}}d\xi \\
&= 2\sqrt{\frac{(2n\pi a)^2 + h^2}{2gh}}.
\end{aligned}
$$

Let's reflect briefly on our answer. What happens as $a \to 0$? Does this seem reasonable to you? What happens as n gets large?

Reference: *Calculus Projects Using Mathematica*, by A. Andrew, G. Cain, S. Crum, and T. Morley

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