Flying in the Wind

Suppose you wish to fly(in an airplane) from point A to point B. In the absence of wind, then you simply fly in the direction of the displacement vector $\mathbf{d}$ from A to B. If the speed of the plane is s miles per hour (m.p.h.), then the trip will take $|\mathbf{d}|/s$ hours. More formally, if the velocity of the plane is $\mathbf{v}$, then the displacement after time t is simply $t\mathbf{v}$, and we want to have $t\mathbf{v} = \mathbf{d}$. Thus, $t = |\mathbf{d}|/|\mathbf{v}| = |\mathbf{d}|/s$, and $\mathbf{v} = (1/t)\mathbf{d}$.

This is pretty simple stuff. The excitement begins when the wind begins to blow. If there is a wind with velocity $\mathbf{w}$, then the displacement of our plane is no longer $t\mathbf{v}$, where $\mathbf{v}$ is the plane's velocity, but it is $t(\mathbf{v}+\mathbf{w})$. Now we need to find t and $\mathbf{v}$ so that $t(\mathbf{v}+\mathbf{w}) = \mathbf{d}$. This vector equation is a bit more difficult to solve. Let's see what we can do.

First, look at the scalar product of $\mathbf{v}$ with the equation $t(\mathbf{v}+\mathbf{w}) = \mathbf{d}$. This gives us

$$t\mathbf{v} \bullet \mathbf{v} + t\mathbf{w} \bullet \mathbf{v} = \mathbf{d} \bullet \mathbf{v}.$$

Next, look at the scalar product of $\mathbf{w}$ with the equation $t(\mathbf{v}+\mathbf{w}) = \mathbf{d}$:

$$t\mathbf{v} \bullet \mathbf{w} + t\mathbf{w} \bullet \mathbf{w} = \mathbf{d} \bullet \mathbf{w}.$$

Substituting the value of $t\mathbf{v} \bullet \mathbf{w}$ from this equation into the first equation gives us

$$t\mathbf{v} \bullet \mathbf{v} + \mathbf{d} \bullet \mathbf{w} - t\mathbf{w} \bullet \mathbf{w} = \mathbf{d} \bullet \mathbf{v}.$$

Multiply this one by t :

$$t^2\mathbf{v} \bullet \mathbf{v} + t\mathbf{d} \bullet \mathbf{w} - t^2\mathbf{w} \bullet \mathbf{w} = t\mathbf{d} \bullet \mathbf{v}.$$

Now scalar multiply the very first equation by $\mathbf{d}$:

$$t\mathbf{v} \bullet \mathbf{d} + t\mathbf{w} \bullet \mathbf{d} = \mathbf{d} \bullet \mathbf{d}, \text{ or}$$

$$t\mathbf{v} \bullet \mathbf{d} = \mathbf{d} \bullet \mathbf{d} - t\mathbf{w} \bullet \mathbf{d}$$

Finally, substitute this into the previous equation:

$$t^2\mathbf{v} \bullet \mathbf{v} + t\mathbf{d} \bullet \mathbf{w} - t^2\mathbf{w} \bullet \mathbf{w} = \mathbf{d} \bullet \mathbf{d} - t\mathbf{w} \bullet \mathbf{d}.$$

Tidying this up, we have

$$t^2(s^2 - w^2) + 2t\mathbf{d} \bullet \mathbf{w} - d^2 = 0,$$

where $s = |\mathbf{v}|$ is the speed of the plane, $w = |\mathbf{w}|$ is the wind speed, and $d = |\mathbf{d}|$ is the distance between A and B. Now the possible solutions for this quadratic will be

$$t = \frac{-\mathbf{d} \bullet \mathbf{w} \pm \sqrt{(\mathbf{d} \bullet \mathbf{w})^2 + d^2(s^2 - w^2)}}{s^2 - w^2}.$$

Let's reflect on this. First, suppose $s > w$; that is, suppose the speed of our aircraft is greater than the wind speed. Then $s^2 - w^2 > 0$, and so the radical term $\sqrt{(\mathbf{d} \bullet \mathbf{w})^2 + d^2(s^2 - w^2)} > |\mathbf{d} \bullet \mathbf{w}|$, giving us exactly one positive solution

$$t = \frac{-\mathbf{d} \bullet \mathbf{w} + \sqrt{(\mathbf{d} \bullet \mathbf{w})^2 + d^2(s^2 - w^2)}}{s^2 - w^2}.$$

Then it's easy to find $\mathbf{v} = \frac{1}{t}\mathbf{d} - \mathbf{w}$, and we are all done.

Let's look at an example. Suppose you are to fly from Atlanta to Charlotte, your plane has a speed of 135 m.p.h., and there is a 35 m.p.h. wind from the Southeast. In this case $d = |\mathbf{d}| = 224$ miles; $s = 135$, $w = 35$ m.p.h., and the compass heading from Atlanta to Charlotte is 63 degrees. The angle φ between $\mathbf{d}$ and $\mathbf{w}$ is thus $\varphi = 45 + 63 = 108$ degrees. Then

$$\mathbf{d} \bullet \mathbf{w} = dw\cos\varphi = (224)(35)\cos((108/180)\pi) = -2422.7.$$

Thus

$$t = \frac{-\mathbf{d} \bullet \mathbf{w} + \sqrt{(\mathbf{d} \bullet \mathbf{w})^2 + d^2(s^2 - w^2)}}{s^2 - w^2}$$
$$= 1.87 \text{ hours}$$

For the heading, we have

$$\mathbf{v} = \frac{1}{t}\mathbf{d} - \mathbf{w} = \frac{1}{1.87}(224\cos(27^o)\mathbf{i} + 224\sin(27^o)\mathbf{j}) - (-35\cos(45^o)\mathbf{i} + 35\sin(45^o)\mathbf{j})$$

$$\mathbf{v} = 131.46\mathbf{i} + 29.633\mathbf{j}$$

Now then $\arctan(29.633/131.46) = 0.22171$, or 12.703 degrees. The heading is thus 90 – 12.703 = 77.297 degrees.

Much more exciting is the case in which the our plane's speed is less than the wind speed: $s^2 - w^2 < 0$. Then the expression under the radical looks like

$$
\begin{aligned}
(\mathbf{d}\bullet\mathbf{w})^2 + d^2(s^2 - w^2) &= d^2w^2\cos^2\varphi - d^2(w^2 - s^2) \\
&= d^2(w^2\cos^2\varphi - w^2 + s^2),
\end{aligned}
$$

where φ is the angle between **d** and **w**. Thus there are real solutions only in case

$$
w^2\cos^2\varphi - w^2 + s^2 \geq 0, \text{ or}
$$

$$
\cos^2\varphi \geq 1 - \left(\tfrac{s}{w}\right)^2.
$$

Next, we need to decide which are positive. Convince yourself that there will be no positive value of t if $\mathbf{d}\bullet\mathbf{w} < 0$. Thus we must have $\cos\varphi \geq 0$. Putting this together with our previous result, we need to have

$$
\cos\varphi \geq \sqrt{1 - \left(\tfrac{s}{w}\right)^2}.
$$

Observe that when this is so, there are *two* positive values of t, giving us two different directions that will take us from A to B. Can you explain what's going on?

Let's try another example. Suppose once again we are to fly from Atlanta to Charlotte in our 135 m.p.h. plane, but now there is a 165 m.p.h. wind from the South. As in the previous example, $d = 224$ and $s = 135$. But now, $w = 165$ and the angle φ between **d** and **w** is $\varphi = 63$ degrees. Then

$$
\varsigma = \arccos\left(\sqrt{1 - \left(\tfrac{s}{w}\right)^2}\right) = \arccos(0.57496) = 57.5 \text{ degrees.}
$$

We see then that it is impossible for us to get to Charlotte in this wind.

Suppose the 165 m.p.h. wind shifts so that it is coming from the Southeast. Then $\varphi = 63 - 45 = 18$ degrees, and we should be able to make it now. Let's see how.

$$
\mathbf{d}\bullet\mathbf{w} = (224)(165)\cos(18(\pi/180)) = 35151
$$

: 35151.

and so

$$
\begin{aligned}
t &= \frac{-\mathbf{d}\bullet\mathbf{w} \pm \sqrt{(\mathbf{d}\bullet\mathbf{w})^2 + d^2(s^2 - w^2)}}{s^2 - w^2} \\
&= \frac{-35151 \pm 28000}{-9000}
\end{aligned}
$$

We see that we have *two* positive values of t:

$$t_1 = \frac{-35151 + 28000}{-9000} = 0.8 \text{ hours, and}$$
$$t_2 = \frac{-35151 - 28000}{-9000} = 7.0 \text{ hours.}$$

Look first at the quick trip, the one which takes a mere 0.8 hours.

$$\begin{aligned}\mathbf{v} &= \frac{1}{0.8}\mathbf{d} - \mathbf{w} \\ &= \frac{1}{0.8}(224\cos(27^o)\mathbf{i} + 224\sin(27^o)\mathbf{j}) - 165(\cos(45^o)\mathbf{i} + \sin(45^o)\mathbf{j}\end{aligned}$$

or,

$$\mathbf{v} = 132.8\mathbf{i} + 10.45\mathbf{j}$$

Now, $\arctan(10.45/132.8) = 4.5$ degrees. Our heading should thus be $90 - 4.5 = 85.5$ degrees.

Let's see what the other possibility looks like. Here we have $t = 7.0$ and so

$$\begin{aligned}\mathbf{v} &= \frac{1}{7.0}\mathbf{d} - \mathbf{w} \\ &= \frac{1}{7.0}(224\cos(27^o)\mathbf{i} + 224\sin(27^o)\mathbf{j}) - 165(\cos(45^o)\mathbf{i} + \sin(45^o)\mathbf{j}\end{aligned}$$

$$\mathbf{v} = -88.16\mathbf{i} - 102.14\mathbf{j}$$

Now, $\arctan(88.16/102.14) = 40.798$, and so the heading is $180 + 40.8 = 220.8$ degrees. Reflect and meditate on these results.

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