Maps, Great Circles, Rhumb Lines, and All That

I. Mercator's Map. The domain of the usual spherical coordinates vector description, or parameterization, of a sphere of radius a

$$\mathbf{r}(\lambda,\varphi) = a(\cos\varphi\cos\lambda\mathbf{i} + \cos\varphi\sin\lambda\mathbf{j} + \sin\varphi\mathbf{k}),$$

where φ is latitude and λ is longitude, provides a map of the surface of the Earth. This map is the *plate carée* and is one of the oldest map projections known. It has essentially nothing to recommend it. It is clear, of course, that shapes are distorted–on the map, meridians are parallel, while on the surface of the Earth, they meet at the poles. Worse yet for navigational purposes, it is not conformal: a straight line on the map does not represent a curve on the Earth that intersects each meridian at the same angle (Such a curve is called a *rhumb line.).* Let's see if we can modify this map to obtain one on which straight lines represent rhumb lines. Specifically, we shall "scale" the vertical axis so that the resulting vector function is conformal. He we go.

The horizontal axis of our domain will remain simply longitude λ. Suppose latitude φ is a to-be-determined function $\varphi(y)$ of the vertical, or y-coordinate. Our vector description of the Earth's surface is thus

$$\mathbf{R}(\lambda,y) = a(\cos\varphi(y)\cos\lambda\mathbf{i} + \cos\varphi(y)\sin\lambda\mathbf{j} + \sin\varphi(y)\mathbf{k})$$

Now, the image of the straight line

$$L(\lambda) = m(\lambda - \lambda_0) + y_0$$

on the map is the curve described by

$$\mathbf{P}(\lambda) = \mathbf{R}(\lambda, L(\lambda)).$$

If γ is the angle between this curve and the meridian passing through $\mathbf{R}(\lambda, L(\lambda))$, then

$$\cos\gamma = \frac{\mathbf{P}'(\lambda)\cdot\frac{\partial\mathbf{R}}{\partial y}}{|\mathbf{P}'(\lambda)|\left|\frac{\partial\mathbf{R}}{\partial y}\right|}.$$

Now,

$$\frac{\partial\mathbf{R}}{\partial y} = a(-\sin\varphi\cos\lambda\mathbf{i} - \sin\varphi\sin\lambda\mathbf{j} + \cos\varphi\mathbf{k})\frac{d\varphi}{dy}, \text{ and}$$

$$\mathbf{P}'(\lambda) = \frac{\partial\mathbf{R}}{\partial\lambda} + \frac{\partial\mathbf{R}}{\partial y}L'(\lambda) = \frac{\partial\mathbf{R}}{\partial\lambda} + \frac{\partial\mathbf{R}}{\partial y}m.$$

Hence,

$$\cos\gamma = \frac{\mathbf{P}'(\lambda)\cdot\frac{\partial\mathbf{R}}{\partial y}}{|\mathbf{P}'(\lambda)|\left|\frac{\partial\mathbf{R}}{\partial y}\right|} = \frac{m\left|\frac{\partial\mathbf{R}}{\partial y}\right|^2}{|\mathbf{P}'(\lambda)|\left|\frac{\partial\mathbf{R}}{\partial y}\right|} = \frac{m\left|\frac{\partial\mathbf{R}}{\partial y}\right|}{|\mathbf{P}'(\lambda)|}.$$

A bit of calculation yields

$$|\mathbf{P}'(\lambda)| = a\sqrt{\cos^2\varphi + m^2\left(\frac{d\varphi}{dy}\right)^2}, \text{and}$$

$$\left|\frac{\partial \mathbf{R}}{\partial y}\right| = a\left|\frac{d\varphi}{dy}\right|.$$

Thus,

$$\cos\gamma = \frac{m\left|\frac{\partial \mathbf{R}}{\partial y}\right|}{|\mathbf{P}'(\lambda)|} = \frac{m}{\sqrt{\cos^2\varphi + m^2\left(\frac{d\varphi}{dy}\right)^2}}\frac{d\varphi}{dy}.$$

The slope of our straight line on the map is m, so the cosine of the angle between this line and the vertical axis is $\frac{m}{\sqrt{1+m^2}}$. We therefore want to have

$$\frac{m}{\sqrt{\cos^2\varphi + m^2\left(\frac{d\varphi}{dy}\right)^2}}\frac{d\varphi}{dy} = \frac{m}{\sqrt{1+m^2}},$$

or,

$$\left(\frac{d\varphi}{dy}\right)^2 = \cos^2\varphi$$

It should be clear that we want $\varphi(y)$ to be an increasing function, and we want $\varphi(0) = 0$. This gives us a very nonlinear initial value problem to solve for $\varphi(y)$:

$$\frac{d\varphi}{dy} = \cos\varphi, \ \varphi(0) = 0.$$

We do have seperable variables. Thus,

$$\ln(\sec\varphi + \tan\varphi) = y + c,$$

and the inital condition becomes $\ln(1) = c = 0$. Hence,

$$\sec\varphi + \tan\varphi = e^y,$$

or, after a bit of trigonometric hocus-pocus,

$$e^y = \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right).$$

At last,

$$\varphi = 2\tan^{-1}e^y - \frac{\pi}{2}$$

A map so constructed is the invention(discovery?) of the famous Flemish mapmaker **Gerardus Mercator** (1512-1594). [Meditate on the fact that the discoverers of calculus, Newton and Leibniz, were born in 1642 and 1647, respectively.]

II. Distances. Let's compute the so-called rhumb line distance between two points on the surface of the Earth—this being the length of the rhumb line joining the two points. We now know the rhumb line is simply the image of the straight line segement between the two places on Mercator's

map. Thus, the segment from (λ_0, y_0) to (λ_1, y_1) is

$$L(\lambda) = m(\lambda - \lambda_0) + y_0,\ \lambda_0 \le \lambda \le \lambda_1,$$

where the slope m is

$$m = \frac{y_1 - y_0}{\lambda_1 - \lambda_2}.$$

We have from the previous section that

$$\begin{aligned} |\mathbf{P}'(\lambda)| &= a\sqrt{\cos^2\varphi + m^2\left(\frac{d\varphi}{dy}\right)^2} \\ &= a\sqrt{\cos^2\varphi + m^2\cos^2\varphi} \\ &= a\sqrt{1+m^2}\,|\cos\varphi|. \end{aligned}$$

Hence, the length D of our curve is

$$\begin{aligned} D &= \int_{\lambda_0}^{\lambda_1} |\mathbf{P}'(\lambda)|d\lambda \\ &= a\sqrt{1+m^2}\int_{\lambda_0}^{\lambda_1} \cos\varphi(L(\lambda))d\lambda. \end{aligned}$$

Next let's see what happens if $m = 0$. In that case, $L(\lambda) = y_0$, and our distances becomes simply

$$D = a\cos y_0(\lambda_1 - \lambda_0),$$

which is exactly what we expect. Turn now to the interesting case in which $m \neq 0$ and change variables in the integral for D. Specifically, let $\xi = L(\lambda)$. Then

$$D = a\frac{\sqrt{1+m^2}}{m}\int_{L(\lambda_0)}^{L(\lambda_1)} \cos\varphi(\xi)d\xi.$$

Now let $\psi = \varphi(\xi)$. Then $d\psi = \frac{d\varphi}{d\xi}d\xi = \cos\psi d\xi$. The above integral is then simply

$$\int_{L(\lambda_0)}^{L(\lambda_1)} \cos\varphi(\xi)d\xi = \int_{\varphi_0}^{\varphi_1} d\psi = \varphi_1 - \varphi_0.$$

Hence,

$$D = a\frac{\sqrt{1+m^2}}{m}(\varphi_1 - \varphi_0)$$

Let's now get m in terms of longitude and latitude. First,

$$y_1 - y_0 = \ln(\sec\varphi_1 + \tan\varphi_1) - \ln(\sec\varphi_0 + \tan\varphi_0)$$

$$= \ln\left[\frac{(\sin\varphi_1 + 1)\cos\varphi_0}{(\sin\varphi_0 + 1)\cos\varphi_1} \right]$$

Thus,

$$m = \frac{1}{\lambda_1 - \lambda_0} \ln\left[\frac{(\sin\varphi_1 + 1)\cos\varphi_0}{(\sin\varphi_0 + 1)\cos\varphi_1} \right].$$

Now we shall compute this distance between the two great port cities of Hamburg, Germany, and Wilmington, N. C. Wilmington is at long 77.55 W, lat 34.13 N, while Hamburg is found at long 9.59 E, lat 53.33 N (brrr...). Making the obvious changes in coordinates, this gives us $(\lambda_0, \varphi_0) = \left(0, \frac{34.13}{360} 2\pi\right) = (0, 0.59568)$ for Wilmington, and for Hamburg we have $(\lambda_1, \varphi_1) = \left(\frac{77.55+9.59}{360} 2\pi, \frac{53.33}{360} 2\pi\right) = (1.5209, 0.93078)$. Thus we have

$$m = \frac{1}{1.5209} \ln\left[\frac{(\sin(0.93078) + 1)\cos(0.59568)}{(\sin(0.59568) + 1)\cos(0.93078)} \right] = 0.30905$$

Now,

$$\begin{aligned} D &= a\frac{\sqrt{1+m^2}}{m}(0.93078 - 0.59568) \\ &= 1.1349a \end{aligned}$$

The radius a of the Earth is 3963.0 miles. The rhumb line distance between these two fine cities is therefore

$$D = (3963.0)(1.1349) = 4497.6 \text{ miles}$$

The great circle distance between two spots on Earth, one at (λ_0, φ_0) and one at (λ_1, φ_1) is simply

$$C = a\cos^{-1}\left(\frac{\mathbf{r}(\lambda_0, \varphi_0) \cdot \mathbf{r}(\lambda_1, \varphi_1)}{a^2} \right)$$

In the case of Hamburg and Wilmington, we have

$$\begin{aligned} \cos^{-1}\left(\frac{\mathbf{r}(\lambda_0, \varphi_0) \cdot \mathbf{r}(\lambda_1, \varphi_1)}{a^2} \right) &= \cos^{-1}\left(\frac{\mathbf{r}(0, 0.59568) \cdot \mathbf{r}(1.5209, 0.93078)}{a^2} \right) \\ &= \cos^{-1}(0.47468) = 1.0762 \end{aligned}$$

Thus the great circle distance between the two is

$$C = (1.0762)(3963.0) = 4265.0 \text{ miles.}$$

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