Math 2401A Final Examination Solutions

1. Find the tangent **T**, the principal normal **N**, and the curvature κ for the curve

$$\mathbf{R}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}, \text{ for } t > 0.$$

$$\mathbf{R}'(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j} = t(\cos t\mathbf{i} + \sin t\mathbf{j})$$

Thus,

$$\frac{ds}{dt} = |\mathbf{R}'(t)| = t, \text{ and so}$$

$$\mathbf{T} = \frac{\mathbf{R}'(t)}{|\mathbf{R}'(t)|} = \cos t\mathbf{i} + \sin t\mathbf{j}.$$

Next,

$$\kappa\mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt}$$
$$= \frac{1}{t}(-\sin t\mathbf{i} + \cos t\mathbf{j})$$

Hence,

$$\kappa = |\kappa\mathbf{N}| = \frac{1}{t}, \text{ and}$$
$$\mathbf{N} = -\sin t\mathbf{i} + \cos t\mathbf{j}.$$

2. Find an equation for the plane tangent to $4x^2 + 9y^2 + 36z^2 = 36$ at the point $(1, 2/3, \sqrt{7}/3)$.

We need a point on the plane and a normal to the plane. We already have a point; it's simply $\mathbf{p} = (1, 2/3, \sqrt{7}/3)$. For a normal **n**,

$$\mathbf{n} = \nabla f, \text{ where } f = 4x^2 + 9y^2 + 36z^2.$$

And so,

$$\nabla f = 8x\mathbf{i} + 18y\mathbf{j} + 72z\mathbf{k}$$

.

At $(1, 2/3, \sqrt{7}/3)$ this becomes

$$\mathbf{n} = 8(1)\mathbf{i} + 18\left(\frac{2}{3}\right)\mathbf{j} + 72\left(\frac{\sqrt{7}}{3}\right)\mathbf{k}$$
$$= 8\mathbf{i} + 12\mathbf{j} + 24\sqrt{7}\,\mathbf{k}.$$

Finally, $\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0$ becomes

$$8(x-1) + 12\left(y - \frac{2}{3}\right) + 24\sqrt{7}\left(z - \frac{\sqrt{7}}{3}\right) = 0.$$

3. The temperature at the point (x, y) on a metal plate is $T(x, y) = x^2 - 4xy + y^2$. A bug on the plate walks arounf the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the bug?

The path of the bug is described by

$$\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j},\ 0 \le t \le 2\pi.$$

The temperature $h(t)$ on the bug's path is thus given by

$$h(t) = T(\mathbf{r}(t)).$$

The highest and lowest temperatures will occur at points where the derivative $h'(t) = 0$, or at $\mathbf{r}(0) = \mathbf{r}(2\pi) = (5, 0)$.

Now,

$$\begin{aligned} h'(t) &= \nabla T \bullet \mathbf{r}'(t) \\ &= (2x - 4y)(-5\sin t) + (-4x + 2y)(5\cos t) \\ &= 50[(\cos t - 2\sin t)(-\sin t) + (-2\cos t + \sin t)(\cos t)] \\ &= 50[-4\cos^2 t + 2] \end{aligned}$$

Thus, $h'(t) = 0$ becoses

$$\begin{aligned} 50[-4\cos^2 t + 2] &= 0 \\ \cos^2 t &= \frac{1}{2}, \text{ and so} \\ \cos t &= \frac{1}{\sqrt{2}}, \text{ and } \cos t = -\frac{1}{\sqrt{2}} \end{aligned}$$

When $\cos t = \frac{1}{\sqrt{2}}$, then $\sin t = \frac{1}{\sqrt{2}}$, or $-\frac{1}{\sqrt{2}}$. Similarly, when $\cos t = \frac{1}{\sqrt{2}}$, then $\sin t = \frac{1}{\sqrt{2}}$, or $-\frac{1}{\sqrt{2}}$. Our candidates for points at which the maximum and minimum occurs are thus

$$\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right), \left(\frac{5}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right), \text{ and } \left(-\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right),$$

together, of course, with $(5, 0)$. Now,

$T\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right) = T\left(-\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right) = -25$

$T\left(\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right) = T\left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right) = 75$, and

$T(5, 0) = 25.$

Thus the highest temperature encountered by the bug is 75, and the lowest is -25.

4. A wire of density $\lambda(x, y, z) = 15\sqrt{y + 2}$ lies along the curve

$$\mathbf{R}(t) = (t^2 - 1)\mathbf{i} + 2t\mathbf{k}, \quad -1 \le t \le t.$$

Find its mass.

The mass M is simply,

$$M = \int_W \lambda(x, y, z)dr$$

First,

$$\int_W \lambda(x,y,z)dr = \int_{-1}^{1} \lambda(\mathbf{R}(t))|\mathbf{R}'(t)|dt$$

.

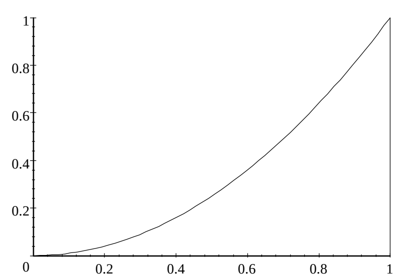
Now, $\lambda(\mathbf{R}(t)) = 15\sqrt{(t^2-1+2)} = 15\sqrt{t^2+1}$, and $\mathbf{R}'(t) = 2t\mathbf{j} + 2\mathbf{k}$. Hence $|\mathbf{R}'(t)| = 2\sqrt{t^2+1}$.

$$\begin{aligned}\int_W \lambda(x,y,z)dr &= \int_{-1}^{1} \lambda(\mathbf{R}(t))|\mathbf{R}'(t)|dt \\ &= \int_{-1}^{1} 15\sqrt{t^2+1}\;2\sqrt{t^2+1}\,dt \\ &= 30\int_{-1}^{1}(t^2+1)dt = 30\frac{8}{3} = 80.\end{aligned}$$

5. Evaluate

$$\int_0^1 \int_{\sqrt{y}}^{1} \sqrt{1+x^3}\,dxdy$$

It is most difficult to find an elementary antiderivative for $\sqrt{1+x^3}$, so we shall reconstruct the two-dimensional integral that gives rise to this iterated integral and see if we can integrate first with respect to y. We must have a picture of the region R :



Now,

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^{1} \sqrt{1+x^3}\,dxdy &= \iint_R \sqrt{1+x^3}\,dA \\ &= \int_0^1 \int_0^{x^2} \sqrt{1+x^3}\,dydx \\ &= \int_0^1 x^2\sqrt{1+x^3}\,dx\end{aligned}$$

$$= \frac{2}{9}(1+x^3)^{3/2}\Big|_0^1 = \frac{2}{9}\left(2\sqrt{2}-1\right)$$

6. What fraction of the Earth's surface lies above latitude 60 degrees North? Explain.

A vector description of the Earth (*i.e.*, a sphere of radius a) is

$$\mathbf{r}(s,t) = a[\cot t \cos s\mathbf{i} + \cos t \sin s\mathbf{j} + \sin t\mathbf{k}],$$

where s is longitude and t is latitude. The entire surface of the sphere is the image of the domain $D = \{(s,t) : 0 \le s \le 2\pi,\ -\pi/2 \le t \le \pi/2\}$, and the part North of latitude 45 North is the image of the domain $N = \{(s,t) : 0 \le s \le 2\pi,\ \pi/3 \le t \le \pi/2\}$. Thus the fraction of the Earth's surface above this latitude is simply

$$\frac{\iint\limits_N d\sigma}{\iint\limits_D d\sigma}$$

Now we need $\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|$:

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = a^2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t \sin s & \cos t \cos s & 0 \\ -\sin t \cos s & -\sin t \sin s & \cos t \end{vmatrix}$$

$$= a^2[\cos^2 t \cos s\mathbf{i} + \cos^2 t \sin s\mathbf{j} + (\cos t \sin t \sin^2 s + \cos t \sin t \cos^2 s)\mathbf{k}$$

$$= a^2 \cos t[\cos t \cos s\mathbf{i} + \cos t \sin s\mathbf{j} + \sin t\mathbf{k}]$$

Thus,

$$\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right| = a^2|\cos t|$$

Hence,

$$\iint\limits_N d\sigma = \int_0^{2\pi}\int_{\pi/3}^{\pi/2} a^2|\cos t| dt ds = a^2\int_0^{2\pi}[\sin t]_{\pi/3}^{\pi/2} ds$$

$$= a^2 2\pi\left[1 - \frac{\sqrt{3}}{2}\right]$$

In the same way (or you may remember the area of a sphere),

$$\iint\limits_D d\sigma = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} a^2 \cos t dt ds = 4\pi a^2.$$

And so we have

$$\frac{\iint\limits_N d\sigma}{\iint\limits_D d\sigma} = \frac{a^2 2\pi\left[1 - \frac{\sqrt{3}}{2}\right]}{4\pi a^2} = \frac{2-\sqrt{3}}{4}$$