Math 2401B1&B2 Final Examination Solutions

1. Find an equation for the plane tangent to $4x^2 + 9y^2 + 36z^2 = 36$ at the point $(1, 2/3, -\sqrt{7}/3)$.

We need a point on the plane and a normal to the plane. We already have a point; it's simply $\mathbf{p} = (1, 2/3, -\sqrt{7}/3)$. For a normal \mathbf{n} ,

n =
$$\nabla f$$
, where $f = 4x^2 + 9y^2 + 36z^2$.

And so,

$$\nabla f = 8x\mathbf{i} + 18y\mathbf{j} + 72z\mathbf{k}$$

At $(1, 2/3, -\sqrt{7}/3)$ this becomes

$$\mathbf{n} = 8(1)\mathbf{i} + 18\left(\frac{2}{3}\right)\mathbf{j} + 72\left(-\frac{\sqrt{7}}{3}\right)\mathbf{k}$$
$$= 8\mathbf{i} + 12\mathbf{j} - 24\sqrt{7}\mathbf{k}.$$

Finally, $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ becomes

$$8(x-1) + 12\left(y - \frac{2}{3}\right) - 24\sqrt{7}\left(z - \frac{\sqrt{7}}{3}\right) = 0.$$

2. Find the tangent **T**, the principal normal **N**, and the curvature κ for the curve

 $\mathbf{R}(t) = 3(\sin t - t\cos t)\mathbf{i} - 4\mathbf{j} + 3(\cos t + t\sin t)\mathbf{k}, \text{ for } t > 0.$

$$\mathbf{R}'(t) = 3t\sin t\mathbf{i} + 3t\cos t\mathbf{k}$$

Thus,

$$\frac{ds}{dt} = |\mathbf{R}'(t)| = 3t, \text{ and so}$$
$$\mathbf{T} = \frac{\mathbf{R}'(t)}{|\mathbf{R}'(t)|} = \sin t\mathbf{i} + \cos t\mathbf{k}$$

Next,

$$\kappa \mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} / \frac{ds}{dt}$$
$$= \frac{1}{3t} (\cos t \mathbf{i} - \sin t \mathbf{k})$$

Hence,

$$\kappa = |\kappa \mathbf{N}| = \frac{1}{3|t|} = \frac{1}{3t}$$
, and
 $\mathbf{N} = \cos t \mathbf{i} - \sin t \mathbf{k}$.

3. The temperature at the point (x, y) on a metal plate is $T(x, y) = 2x^2 - 8xy + 2y^2$. A bug on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the bug?

The path of the bug is described by

 $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}, \ 0 \le t \le 2\pi.$

The temperature h(t) on the bug's path is thus given by

$$h(t) = T(\mathbf{r}(t)).$$

The highest and lowest temperatures will occur at points where the derivative h'(t) = 0, or at $\mathbf{r}(0) = \mathbf{r}(2\pi) = (5,0)$.

Now,

$$h'(t) = \nabla T \bullet \mathbf{r}'(t)$$

= $(4x - 8y)(-5\sin t) + (-8x + 4y)(5\cos t)$
= $100[(\cos t - 2\sin t)(-\sin t) + (-2\cos t + \sin t)(\cos t)]$
= $100[-4\cos^2 t + 2]$

Thus, h'(t) = 0 becomes

$$100[-4\cos^{2}t + 2] = 0$$

$$\cos^{2}t = \frac{1}{2}, \text{ and so}$$

$$\cos t = \frac{1}{\sqrt{2}}, \text{ and } \cos t = -\frac{1}{\sqrt{2}}$$

When $\cos t = \frac{1}{\sqrt{2}}$, then $\sin t = \frac{1}{\sqrt{2}}$, or $-\frac{1}{\sqrt{2}}$. Similarly, when $\cos t = \frac{1}{\sqrt{2}}$, then $\sin t = \frac{1}{\sqrt{2}}$, or $-\frac{1}{\sqrt{2}}$. Our candidates for points at which the maximum and minimum occurs are thus

$$\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right), \left(\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right), \left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right), \text{and} \left(-\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right)$$

together, of course, with (5,0). Now,

$$T\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right) = T\left(-\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right) = -50$$
$$T\left(\frac{5}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right) = T\left(-\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}\right) = 150, \text{ and}$$
$$T(5, 0) = 50.$$

Thus the highest temperature encountered by the bug is 150, and the lowest is -50.

4. A wire of density $\lambda(x, y, z) = \sqrt{x+2}$ lies along the curve

$$\mathbf{R}(t) = (t^2 - 1)\mathbf{i} + 2t\mathbf{k}, \quad -1 \le t \le t.$$

Find its mass.

The mass *M* is simply,

$$M = \int_{W} \lambda(x, y, z) dr$$

First,

$$\int_{W} \lambda(x, y, z) dr = \int_{-1}^{1} \lambda(\mathbf{R}(t)) |\mathbf{R}'(t)| dt$$

Now, $\lambda(\mathbf{R}(t)) = \sqrt{(t^2 - 1 + 2)} = \sqrt{t^2 + 1}$, and $\mathbf{R}'(t) = 2t\mathbf{i} + 2\mathbf{k}$. Hence $|\mathbf{R}'(t)| = 2\sqrt{t^2 + 1}$.

$$\int_{W} \lambda(x, y, z) dr = \int_{-1}^{1} \lambda(\mathbf{R}(t)) |\mathbf{R}'(t)| dt$$
$$= \int_{-1}^{1} \sqrt{t^2 + 1} \ 2\sqrt{t^2 + 1} \ dt$$
$$= 2 \int_{-1}^{1} (t^2 + 1) dt = 2\frac{8}{3} = \frac{16}{3}.$$

5. Evaluate

$$\int_{0}^{1} \int_{\sqrt{x}}^{1} \sqrt{3 + 2y^3} \, dy dx$$

It is most difficult to find an elementary antiderivative for $\sqrt{3+2y^3}$, so we shall reconstruct the two-dimensional integral that gives rise to this iterated integral and see if we can integrate first with respect to *x*. We must have a picture of the region *R* :



Now, holding *y* fixed and integrating first with respect to *x*, we have

$$\int_{0}^{1} \int_{\sqrt{x}}^{1} \sqrt{3 + 2y^{3}} \, dy \, dx = \iint_{R} \sqrt{3 + 2y^{3}} \, dA = \int_{0}^{1} \int_{0}^{y^{2}} \sqrt{3 + 2y^{3}} \, dx \, dy$$
$$= \int_{0}^{1} y^{2} \sqrt{3 + 2y^{3}} \, dy = \frac{(3 + 2y^{3})^{3/2}}{9} \Big|_{0}^{1} = \frac{1}{9} \left(5\sqrt{5} - 3\sqrt{3} \right)$$

6. What fraction of the Earth's surface lies above latitude 45 degrees North? Explain.

A vector description of the Earth (*i.e.*, a sphere of radius *a*) is

 $\mathbf{r}(s,t) = a[\cot t \cos s\mathbf{i} + \cos t \sin s\mathbf{j} + \sin t\mathbf{k}],$

where *s* is longitude and *t* is latitude. The entire surface of the sphere is the image of the domain $D = \{(s,t) : 0 \le s \le 2\pi, -\pi/2 \le t \le \pi/2\}$, and the part North of latitude 45 North is the image of the domain $N = \{(s,t) : 0 \le s \le 2\pi, \pi/4 \le t \le \pi/2\}$. Thus the fraction of the Earth's surface above this latitude is simply

$$\frac{\iint_N d\sigma}{\iint_D d\sigma}$$

Now we need $\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right|$:

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = a^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t \sin s & \cos t \cos s & 0 \\ -\sin t \cos s & -\sin t \sin s & \cos t \end{vmatrix}$$
$$= a^2 [\cos^2 t \cos s \mathbf{i} + \cos^2 t \sin s \mathbf{j} + (\cos t \sin t \sin^2 s + \cos t \sin t \cos^2 s) \mathbf{k}]$$
$$= a^2 \cos t [\cos t \cos s \mathbf{i} + \cos t \sin s \mathbf{j} + \sin t \mathbf{k}]$$

Thus,

$$\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right| = a^2 |\cos t|$$

Hence,

$$\iint_{N} d\sigma = \int_{0}^{2\pi} \int_{\pi/4}^{\pi/2} a^{2} |\cos t| dt ds = a^{2} \int_{0}^{2\pi} [\sin t]_{\pi/4}^{\pi/2} ds$$
$$= a^{2} 2\pi \left[1 - \frac{1}{\sqrt{2}} \right] = \pi a^{2} (2 - \sqrt{2})$$

In the same way (or you may remember the area of a sphere),

$$\iint_D d\sigma = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} a^2 \cos t dt ds = 4\pi a^2.$$

And so we have

$$\frac{\iint_{N} d\sigma}{\iint_{D} d\sigma} = \frac{\pi a^{2}(2-\sqrt{2})}{4\pi a^{2}} = \frac{2-\sqrt{2}}{4}.$$