Math 2401 Quiz 3 Solutions

1. The directional derivative in the direction of a unit vector **u** is simply the scalar, or dot, product (grad f) $\bullet$ **u** $= |\text{grad } f||\mathbf{u}|\cos\varphi = |\text{grad } f|\cos\varphi$, where φ is the angle between the gradient grad f and the direction. For $0 \le \varphi \le \pi$, the cosine is a decreasing function of φ. Hence the larger the angle between the gradient and the direction, the smaller the directional derivative. Thus we have: $a > d > c > b$.

2. Find all points on the surface $x + y + z + xy - x^2 - y^2 = 0$ at which the tangent plane is parallel to the plane $8x - 10y - 2z = 73$.

At the point (x_0, y_0, z_0), the direction normal to the surface is the direction normal to the tangent plane. For the tangent plane to be parallel to the plane $8x - 10y - 2z = 73$, the normal direction of the tangent plane must be the direction normal to the plane. This is, of course, the direction of the vector $8\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}$. A vector normal to the surface at (x, y, z) is simply the gradient

$$\nabla(x + y + z + xy - x^2 - y^2) = (1 + y - 2x)\mathbf{i} + (1 + x - 2y)\mathbf{j} + \mathbf{k}$$

We thus need to find (x, y, z) so that

$$(1 + y - 2x)\mathbf{i} + (1 + x - 2y)\mathbf{j} + \mathbf{k} = \boldsymbol{\alpha}(8\mathbf{i} - 10\mathbf{j} - 2\mathbf{k})$$

for some scalar $\alpha \neq 0$. This leads to the system of equations

$$\begin{aligned} 1 + y - 2x &= 8\alpha \\ 1 + x - 2y &= -10\alpha \\ 1 &= -2\alpha \\ x + y + z + xy - x^2 - y^2 &= 0 \end{aligned}$$

This is pretty easy to solve. Note first from the third equation that we must have $\alpha = -\frac{1}{2}$.
The first two equations now become

$$\begin{aligned} 1 + y - 2x &= -4 \\ 1 + x - 2y &= 5 \end{aligned}$$

or,

$$\begin{aligned} -2x + y &= -5 \\ x - 2y &= 4 \end{aligned}$$

There is exactly one solution: $x = 2$, $y = -1$. Finally, from the last equation,

$$2 - 1 + z - 2 - 4 - 1 = 0, \text{ or}$$
$$z = 6$$

In summary, the tangent plane to the given surface is parallel to the given plane at exactly one point: $(2, -1, 6)$.

3. A particle is moving through space. Explain how we know that at the instant it passes through the coldest point on its trajectory, the velocity is perpendicular to the temperature gradient.

Let $T(x,y,z)$ be the temperature at the point (x,y,z), and let $\mathbf{r}(t)$ be the position of the particle at time t. Then the temperature $f(t)$ on the trajectory at time t is simply

$$f(t) = T(\mathbf{r}(t)).$$

At the time t_m at which the minimum of f occurs, we know we must have $f'(t_m) = 0$, or, using the chain rule,

$$f'(t) = \nabla T(\mathbf{r}(t_m) \bullet \mathbf{r}'(t_m) = 0$$

Hence, the vectors $\nabla T(\mathbf{r}(t_m)$ and $\mathbf{r}'(t_m) = \mathbf{v}(t_m)$ are perpendicular.

4. The temperature of the point (x,y) on the rectangular plate $0 \le x \le 1$, $0 \le y \le 1$, is given by $T(x,y) = 48xy - 32x^3 - 24y^2$. Find the hottest and coldest points on the plate.

First, look for places in the interior at which an extreme value of T can occur by seeing where the gradient, $\nabla T = 0$.

$$\nabla T = (48y - 96x^2)\mathbf{i} + (48x - 48y)\mathbf{j} = 0.$$

Thus we have the system

$$48y - 96x^2 = 0$$
$$48x - 48y = 0$$

From the second of these, we know $x = y$. Putting this into the first equation, we have

$$48x - 96x^2 = 0, \text{ or}$$
$$48x(1-2x) = 0.$$

The only solution in the interior of our given region is $x = 1/2$, and $y = 1/2$.

Next, let's look on the boundary of the rectangle. First, on the bottom edge, where $y = 0$. A vector description of this side is simply $\mathbf{r}(t) = t\mathbf{i}$, for $0 \le t \le 1$. Thus

$$T(\mathbf{r}(t)) = -32t^3.$$

Ones sees at once there are no places where an extreme value of this can occur except at the ends: $(0,0)$ and $(1,0)$. Next, the top edge, $\mathbf{r}(t) = t\mathbf{i} + \mathbf{j}, 0 \le t \le 1$. Here we have

$$T(\mathbf{r}(t)) = 48t - 32t^3 - 24 = 8(-4t^3 + 6t - 3)$$

Now,

$$\frac{d}{dt}T(\mathbf{r}(t)) = 8(-12t^2 + 6).$$

The sole place inside the interval $[0,1]$ at which this derivative vanishes is $t = 1/\sqrt{2}$. And,as usual, the end points are candidates for extreme values. Thus we have possibilities $(0,1)$, $(1/\sqrt{2},1)$, and $(1,1)$.

Continuing, we look at the left side, $\mathbf{r}(t) = t\mathbf{j}, 0 \le t \le 1$. Here, we have

$$T(\mathbf{r}(t)) = -24t^2,$$

and we find no additional places of interest. Finally, the right side: $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j},\ 0 \leq t \leq 1$. Here

$$T(\mathbf{r}(t)) = 48t - 24t^2 = 24(2t - t^2).$$

Then,

$$\frac{d}{dt}T(\mathbf{r}(t)) = 24(2 - 2t),$$

and we have no additional points of interest. We now simply find the temperature at all the possibilities we found:

$$\begin{aligned} T(1/2, 1/2) &= 2 \\ T(0,0) &= 0 \\ T(1,0) &= -32 \\ T(0,1) &= -24 \\ T(1/\sqrt{2}, 1) &= 16\sqrt{2} - 24 \\ T(1,1) &= -8 \end{aligned}$$

It is now easy to see that $(1/2, 1/2)$ is the hottest point on the plate, and $(1, 0)$ is the coldest.

Here is a picture:

$T(x,y)$