Math 2401 Quiz Four Solutions

1. Find the minimum distance between the curve $y = x^2$ and the point $(0,a)$, where we know that $a > 0$.

We simply want the minimum of $f(x,y) = x^2 + (y-a)^2$ subject to the constraint $g(x,y) = y - x^2 = 0$. Using the method of Lagrange, we want to solve

$$\nabla f = \lambda \nabla g$$
$$g = 0$$

Or,

$$2x = -2x\lambda$$
$$2(y-a) = \lambda$$
$$y = x^2$$

From the first of these equations, we can have $x = 0$. In this case, the last equation tells us that $y = 0$, also. Thus, $(0,0)$ is a possibility.

Next, suppose $x \neq 0$. Then the first equation tells us that $\lambda = -1$. Substituting this into the second equation yields $2(y-a) = -1$, or $y = a - \frac{1}{2}$. Now the last equation tells us that there are no solutions if $y = a - \frac{1}{2} < 0$, or if $a < \frac{1}{2}$. Thus, for $a < \frac{1}{2}$, there is only one possibility, $(0,0)$. For values of $a < \frac{1}{2}$, this is the closest point, and so the distance between the curve and the point $(0,a)$ is $\sqrt{f(0,0)} = a$.(Remember, $f(x,y)$ is the square of the distance!).

Now, for $a \geq \frac{1}{2}$, the third equation tells us that $x = \sqrt{a - \frac{1}{2}}$, and $x = -\sqrt{a - \frac{1}{2}}$. So, when $a \geq \frac{1}{2}$, we have three possible closest points , $(0,0)$, $(\sqrt{a - \frac{1}{2}}, a - \frac{1}{2})$, and $(\sqrt{a - \frac{1}{2}}, a - \frac{1}{2})$. We need to substitute back into $f(x,y)$ to see which gives the smallest value:

$$f(0,0) = a^2$$
$$f(\sqrt{a - \frac{1}{2}}, a - \frac{1}{2}) = a - \frac{1}{2} + \frac{1}{4} = a - \frac{1}{4}$$
$$f(-\sqrt{a - \frac{1}{2}}, a - \frac{1}{2}) = a - \frac{1}{4}$$

For $a \geq \frac{1}{2}$, we know that $a - \frac{1}{4} \leq a^2$. Thus the minimum distance in this case is $\sqrt{a - \frac{1}{4}}$.

2. The trajectory of a particle in space is described by
$\mathbf{r}(t) = (\cos t - \frac{1}{2})\mathbf{i} + (\sin t - \frac{1}{2})\mathbf{j} + 10\mathbf{k}$, and the temperature in space is given by
$T(x,y,z) = x^2 + y^2 + z^2$. Find the hottest point on the trajectory.

The temperature of the particle is given by $T(\mathbf{r}(t))$. To find possibilities for a minimum, we look at all places at which $\frac{d}{dt}T(\mathbf{r}(t)) = 0$. Thus,

$$\frac{d}{dt}T(\mathbf{r}(t)) = \nabla T(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = 0.$$

Now,

$$\nabla T = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k},$$

and so

$$\nabla T(\mathbf{r}(t)) \bullet \mathbf{r}'(t) = \left[2(\cos - \frac{1}{2})\mathbf{i} + 2(\sin t - \frac{1}{2})\mathbf{j} + 20\mathbf{k} \right] \bullet (-\sin t\mathbf{i} + \cos t\mathbf{j}) = 0$$

$$-\cos t \sin t + \frac{\sin t}{2} + \sin t \cos t - \frac{\cos t}{2} = 0$$

$$\cos t = \sin t$$

. This happens when $\tan t = 1$, 0r $t = \frac{\pi}{4} + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$. Our candidates for points at which the minimum occurs are thus $\mathbf{r}(\frac{\pi}{4})$ and $\mathbf{r}(\frac{\pi}{4} + \pi) = \mathbf{r}(\frac{5\pi}{4})$.

Now,

$$\mathbf{r}(\frac{\pi}{4}) = \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right)\mathbf{i} + \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right)\mathbf{j} + 10\mathbf{k}$$

and,

$$\mathbf{r}(\frac{5\pi}{4}) = \left(-\frac{1}{\sqrt{2}} - \frac{1}{2} \right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}} - \frac{1}{2} \right)\mathbf{j} + 10\mathbf{k}$$

Finally,

$$\begin{aligned} T(\mathbf{r}(\frac{\pi}{4})) &= \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right)^2 + \left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right)^2 + 10^2 \\ &= 2\left(\frac{3}{4} - \frac{1}{2}\sqrt{2} \right) + 100 \end{aligned}$$

and

$$\begin{aligned} T(\mathbf{r}(\frac{5\pi}{4})) &= \left(-\frac{1}{\sqrt{2}} - \frac{1}{2} \right)^2 + \left(-\frac{1}{\sqrt{2}} - \frac{1}{2} \right)^2 + 10^2 \\ &= 2\left(\frac{3}{4} + \frac{1}{2}\sqrt{2} \right) + 100 \end{aligned}$$

It's clear that $T(\mathbf{r}(\frac{5\pi}{4})) > T(\mathbf{r}(\frac{\pi}{4}))$, and so

$$\mathbf{r}(\frac{5\pi}{4}) = \left(-\frac{1}{\sqrt{2}} - \frac{1}{2} \right)\mathbf{i} + \left(-\frac{1}{\sqrt{2}} - \frac{1}{2} \right)\mathbf{j} + 10\mathbf{k}$$

is the hottest point on the trajectory.

3. Find the Taylor polynomial at $(0,0)$ of degree ≤ 2 for the function $f(x,y) = e^{2y}\sin x$.

We know that

$$p(h,k) = f(0,0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f \Big|_{(0,0)} + \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f \Big|_{(0,0)}$$

Let's compute all these partial derivatives:

$$\frac{\partial f}{\partial x} = e^{2y}\cos x,\ \frac{\partial f}{\partial y} = 2e^{2y}\sin x,$$

$$\frac{\partial^2 f}{\partial x^2} = -e^{2y}\sin x, \quad \frac{\partial^2 f}{\partial x \partial y} = 2e^{2y}\cos x, \quad \frac{\partial^2 f}{\partial y^2} = 4e^{2y}\sin x$$

Now, evaluated at $(0, 0)$, these become:

$$\frac{\partial f}{\partial x} = 1, \text{ and } \frac{\partial^2 f}{\partial x \partial y} = 2.$$

All the others are zero. Thus,

$$\begin{aligned} p(h,k) &= h + \frac{1}{2}(2 \bullet 2hk) \\ &= h + 2hk. \end{aligned}$$

4. The average value of $f(x,y)$ on the region A is 20, and the average of $g(x,y)$ on A is -4. If $\iint\limits_A f(x,y)dA = 7$, then what is $\iint\limits_A g(x,y)dA$? Explain.

If $\overline{X}$ is the average of $f(x,y)$ on A, then

$$\overline{X} = \frac{\iint\limits_A f(x,y)dA}{\text{area of } A}, \text{ or}$$

$$20 = \frac{7}{\text{area of } A}.$$

Thus, area of $A = \frac{7}{20}$.

Next, we know the average of $g(x,y)$ on A is -4. Thus,

$$-4 = \frac{\iint\limits_A g(x,y)dA}{\text{area of } A} = \frac{\iint\limits_A g(x,y)dA}{7/20}.$$

Hence,

$$\iint\limits_A g(x,y)dA = (-4)\frac{7}{20} = -\frac{7}{5}.$$