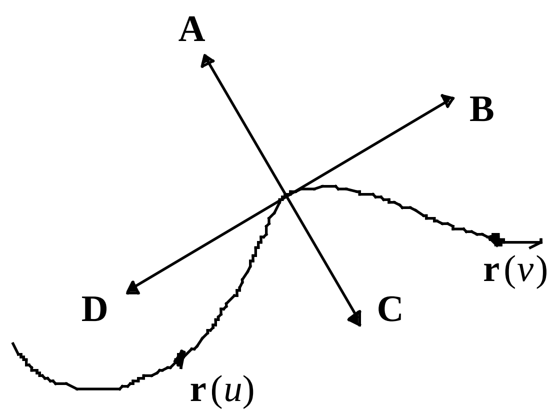
Math 2401 - Quiz One - Solutions

1. In the picture, the curve is described by the vector function $\mathbf{r}(t)$. You may assume that $\mathbf{r}'(t) \neq 0$ for all values of t. The vector $\mathbf{B}$ is tangent to the curve and the vector $\mathbf{A}$ is perpendicular to $\mathbf{B}$. It is also true that $\mathbf{C} = -\mathbf{A}$ and $\mathbf{D} = -\mathbf{B}$. Moreover each of the vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ has length one.

a)Suppose $u < v$. Which of the vectors shown is the unit tangent $\mathbf{T}$ and which is the principal normal $\mathbf{N}$?

b) Suppose $v < u$. Which of the vectors shown is the unit tangent $\mathbf{T}$ and which is the principal normal $\mathbf{N}$?



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a) For $u < v$, we are moving in the direction of $\mathbf{B}$, so in this case the unit tangent $\mathbf{T} = \mathbf{B}$. The tangent is changing in the direction of $\mathbf{C}$, and so the principal normal $\mathbf{N} = \mathbf{C}$.

b) In this case, $\mathbf{T} = \mathbf{D}$ because of the change in direction, but it is still the case that $\mathbf{N} = \mathbf{C}$.

2. A bug starts at the origin and crawls along the curve

$$\mathbf{r}(t) = \frac{2}{3}t^{3/2}\,\mathbf{i} + 3t\mathbf{j} - 4t\mathbf{k}$$

at a constant speed of 5 inches/minute. Find the coordinates of the bug 10 minutes after it starts.

[Give ***exact*** answers, not decimal or other approximations!]

The bug travels a distance of 5(10)=50 inches. The length of the curve from $t = 0$ to $t = a$ is given by

$$L = \int_0^a \|\mathbf{r}'(t)\|dt = \int_0^a \left\|t^{1/2}\,\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}\right\|dt$$

$$= \int_0^a \sqrt{t + 9 + 16}\,dt = \int_0^a \sqrt{t + 25}\,dt = \frac{2}{3}(t + 25)^{3/2}\Big|_0^a$$

$$= \frac{2}{3}\left[(a + 25)^{3/2} - 125\right]$$

The bug will thus end up at $\mathbf{r}(a)$, where a is the solution of

$$L = \frac{2}{3}\left[(a + 25)^{3/2} - 125\right] = 50.$$

Thus,

$$(a + 25)^{3/2} = \frac{3}{2}50 + 125 = 200,$$

and so

$$a = 200^{2/3} - 25.$$

The position of our bug is therefore

$$\mathbf{r}(a) = \frac{2}{3}(200^{2/3} - 25)^{3/2}\,\mathbf{i} + 3(200^{2/3} - 25)\mathbf{j} - 4(200^{2/3} - 25)\mathbf{k}$$

3. Suppose the direction of the only force on a particle is always toward the origin. At time $t = 0$ the particle is at the point $(1,1,1)$ and the initial velocity $\mathbf{v}_0 = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Show the trajectory $\mathbf{r}(t)$ lies in a plane, and find an equation for this plane.

As usual, let $\mathbf{r}$(t) denote the position of the particle and $\mathbf{v} = \mathbf{r}'$, the velocity.
First,

$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{r}\times\mathbf{v}' + \mathbf{r}'\times\mathbf{v} = \mathbf{r}\times\mathbf{v}' + \mathbf{v}\times\mathbf{v} = \mathbf{r}\times\mathbf{v}' .$$

We know from Newton that the force $\mathbf{f}$ is the mass times the acceleration :

$$m\mathbf{v}' = \mathbf{f}$$

But we are given that the force has the direction of $-\mathbf{r}$, and so

$$\mathbf{v}' = \alpha\mathbf{r}.$$

Thus $\mathbf{r}\times\mathbf{v}' = \alpha\mathbf{r}\times\mathbf{r} = 0$, and so $\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = 0$, which tells us that $\mathbf{r}\times\mathbf{v} = \mathbf{c}$, a constant vector. Thus, $\mathbf{r}$ is always perpendicular to the vector $\mathbf{c}$ and so the trajectory lies in the plane passing through the origin that is perpendicular to $\mathbf{c}$.

To find $\mathbf{c}$, compute $\mathbf{r}\times\mathbf{v}$ at time $t = 0$:

$$\mathbf{r}\times\mathbf{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = -\mathbf{i} + \mathbf{k}.$$

An equation of the plane is therefore $-x + z = 0$.

4. Find all points at which the curve

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

intersects the plane $4x + 2y + z = 24$, and convince me that you have found them all.

This one is the first part of Exercise 33, page 793.
We need to find t so that

$$4t + 2t^2 + t^3 = 24$$

Or,

$$t^3 + 2t^2 + 4t - 24 = 0.$$

Simply factor to get

$$(t-2)(t^2 + 4t + 12) = 0$$

One solution is thus $t=2$, and the quadratic formula will tell you that $t^2 + 4t + 12 = 0$ has no solutions. Hence, the only point of intersection is

$$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}.$$