Math* 2401 *Quiz Five Solutions

1. Suppose $\iint_{\Omega} 3dA = 7$, and suppose the average value of $xe^{xy} + \sin(x^2 + y)$ on Ω is 41. Find $\iint_{\Omega} [xe^{xy} + \sin(x^2 + y)]dA$, or explain carefully why it is not possible to do so from the information provided.

We know the average value of $xe^{xy} + \sin(x^2 + y)$ on Ω is given by

$$\frac{1}{\text{Area of }\Omega} \iint_{\Omega} [xe^{xy} + \sin(x^2 + y)]dA = 41.$$

Or,

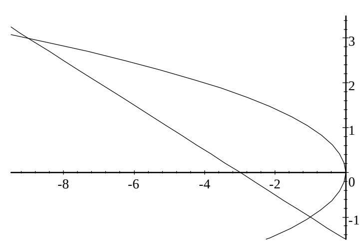
$$\iint_{\Omega} [xe^{xy} + \sin(x^2 + y)]dA = 41(\text{Area of }\Omega).$$

But the area of Ω is $\iint_{\Omega} dA$, and we have $\iint_{\Omega} 3dA = 3\iint_{\Omega} dA = 7$. Thus, $\iint_{\Omega} dA = \frac{7}{3}$, and so

$$\iint_{\Omega} [xe^{xy} + \sin(x^2 + y)]dA = (41)\frac{7}{3} = \frac{287}{3}.$$

2. Find $\iint_{\Omega} ydA$, where Ω is the region bounded by $x + 2y = -3$ and $x = -y^2$.

First, a picture of Ω :



Let's see where these two curves intersect. This is where

$$x = -y^2 = -3 - 2y,$$

or where

$$y^2 - 2y - 3 = (y - 3)(y + 1) = 0.$$

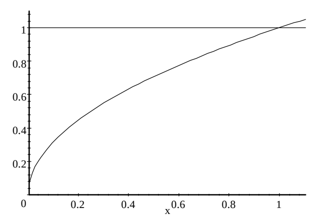
The two intersection points are thus $(-9, 3)$ and $(-1, -1)$.
Now, then

$$\iint_\Omega y\,dA = \int_{-1}^{3}\int_{-3-2y}^{-y^2} y\,dx\,dy = \int_{-1}^{3} y(-y^2+3+2y)\,dy$$

$$= \int_{-1}^{3}(-y^3+3y+2y^2)\,dy = \frac{32}{3}$$

3. Find $\int_0^1\int_{\sqrt{x}}^1 \sqrt{1+y^3}\,dydx$.

I cannot think of an antiderivative for $\sqrt{1+y^3}$, so let's find the two-dimensional integral that lead to this iterated integral and see if changing the order of integration helps.
A picture:



Now,

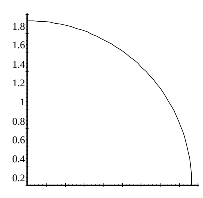
$$\int_0^1\int_{\sqrt{x}}^1 \sqrt{1+y^3}\,dydx = \int_0^1\int_0^{y^2} \sqrt{1+y^3}\,dxdy$$

$$= \int_0^1 y^2\sqrt{1+y^3}\,dy$$

$$= \frac{2}{9}(1+y^3)^{3/2}\Big|_0^1 = \frac{2}{9}(2^{3/2}-1)$$

4. Find the centroid of the part of the circular region $x^2+y^2 \le 3$ that lies in the first quadrant.

A picture:



The centroid (x_c, y_c) is given by

$$x_c = \frac{\iint\limits_{\Omega} x dA}{\iint\limits_{\Omega} dA}, \text{ and } y_c = \frac{\iint\limits_{\Omega} y dA}{\iint\limits_{\Omega} dA}$$

where, of course, Ω is the region of which we are to find the centroid.
First, using polar coordinates,

$$\iint\limits_{\Omega} x dA = \int_0^{\pi/2} \int_0^{\sqrt{3}} r^2 \cos\theta dr d\theta = \int_0^{\pi/2} \cos\theta \frac{r^3}{3} \Big|_0^{\sqrt{3}} d\theta$$

$$= \sqrt{3} \int_0^{\pi/2} \cos\theta d\theta = \sqrt{3}$$

It is easy to see, either by symmetry or direct computation, that $\iint\limits_{\Omega} y dA = \sqrt{3}$, also.

Now, the area of Ω, $\iint\limits_{\Omega} dA$ is simply $\frac{\pi(\sqrt{3})^2}{4} = \frac{3\pi}{4}$. Thus

$$x_c = y_c = \frac{\sqrt{3}}{\frac{3\pi}{4}} = \frac{4}{\pi\sqrt{3}}.$$