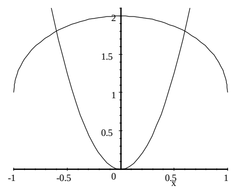
Math 2401 Quiz 6 Solutions

1. Give one iterated integral for the integral $\iint_\Omega x\sin(xy)dA$, where Ω is the part of the region inside the circle $x^2+(y-1)^2 = 1$ that is above the curve $y = 5x^2$. Also give a **Maple** command for evaluating the iterated integral.

As always, begin with a picture:



Next, let's see where the two curves intersect. If $y = 5x^2$ and $x^2+(y-1)^2 = 1$, simply substitute $x^2 = y/5$ into the second of these equations:

$$x^2+(y-1)^2 = \frac{y}{5}+(y-1)^2 = 1, \text{ or}$$

$$\frac{y}{5}+y^2-2y+1 = 1,$$

$$y^2-\frac{9}{5}y = y(y-\frac{9}{5}) = 0$$

The curves thus meet where $y = 0$ (obviously) and where $y = \frac{9}{5}$. Now when $y = \frac{9}{5}$, we have $x^2 = y/5 = \frac{9}{25}$. The two interesting intersection points are thus $(-\frac{3}{5},\frac{9}{5})$ and $(\frac{3}{5},\frac{9}{5})$

Now,

$$\iint_\Omega x\sin(xy)dA = \int_{-3/5}^{3/5}\int_{5x^2}^{1+\sqrt{1-x^2}} x\sin(xy)dydx.$$

A **Maple** command to compute this:

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int(int(x*sin(x*y),y=5*x^2..1+sqrt(1-x^2)),x=-3/5..3/5);
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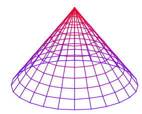
2. Consider the integral

$$\iiint_T (xy+z^2)dV,$$

where T is the solid bounded below by $z = -2$ and above by $z = -\sqrt{x^2+y^2}$.

a)Give an iterated integral for this in which the first integration is with respect to z.

As always, a picture:



The projection of this onto the *x*-*y* plane is simple the circular region *C*: $\sqrt{x^2+y^2} \leq 2$. Thus

$$\iiint_T (xy+z^2)dV = \iint_C \left(\int_{-2}^{-\sqrt{x^2+y^2}} (xy+z^2)dz \right) dA$$

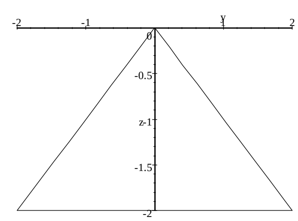
$$= \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{-2}^{-\sqrt{x^2+y^2}} (xy+z^2)dzdxdy, \text{ or}$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-2}^{-\sqrt{x^2+y^2}} (xy+z^2)dzdydx, \text{ or}$$

$$= \int_{0}^{2\pi} \int_{0}^{2} \int_{-2}^{-r} (r^2\cos\theta\sin\theta + z^2)rdzdrd\theta$$

b)Now give an iterated integral in which we integrate first with respect to *x*.

Here we project onto the *y*-*z* (or $x = 0$) plane, and we see $z = -\sqrt{y^2} = |y|$:



Thus,

$$\iiint_T (xy+z^2)dV = \iint_R \left(\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (xy+z^2)dx \right) dA$$

$$= \int_{-2}^{0} \int_{z}^{-z} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} (xy+z^2)dxdydz$$

3. Let $\mathrm{F}(x,y,z) = x\mathbf{i} + (x+y)\mathbf{j} + \mathbf{k}$.

a)Find the integral of **F** from $(1,0,0)$ to $(1,0,2\pi)$ over the path

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi.$$

$\mathbf{F}(\mathbf{r}(t)) = \cos t\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k},$
$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}.$

$$\begin{aligned}\int_P \mathbf{F} \bullet d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt \\ &= \int_0^{2\pi} (-\sin t\cos t + \cos^2 t + \cos t\sin t + 1)dt \\ &= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t)dt + \int_0^{2\pi} dt \\ &= \pi + 2\pi = 3\pi.\end{aligned}$$

b)Find the integral of **F** along the straight line segment joining $(1,0,0)$ and $(1,0,2\pi)$.

$\mathbf{r}(t) = \mathbf{i} + t\mathbf{k},\ 0 \le t \le 2\pi$. Thus, $\mathbf{F}(\mathbf{r}(t)) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{r}'(t) = \mathbf{k}$. Hence,

$$\begin{aligned}\int_L \mathbf{F} \bullet d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

4. Find a function g such that

$$\nabla g = (yz + 2xy + y^2)\mathbf{i} + (x^2 + xz + 2xy + 1)\mathbf{j} + (2z + xy)\mathbf{k}$$

$$\frac{\partial g}{\partial x} = yz + 2xy + y^2, \text{ and so}$$
$$g = xyz + x^2y + xy^2 + \varphi(y,z)$$

Now then

$$\frac{\partial g}{\partial y} = xz + x^2 + 2xy + \frac{\partial \varphi}{\partial y}(y,z) = x^2 + xz + 2xy + 1, \text{ or}$$

$$\frac{\partial \varphi}{\partial y}(y,z) = 1, \text{ and so}$$
$$\varphi(y,z) = y + \lambda(z).$$

Thus,

$$g = xyz + x^2y + xy^2 + \varphi(y,z) = xyz + x^2y + xy^2 + y + \lambda(z).$$

Now,

$$\frac{\partial g}{\partial z} = xy + \lambda'(z) = 2z + xy, \text{ or}$$
$$\lambda'(z) = 2z.$$

Thus, $\lambda(z) = z^2$, and we have

$$\begin{aligned} g &= xyz + x^2y + xy^2 + y + \lambda(z) \\ &= xyz + x^2y + xy^2 + y + z^2 \end{aligned}$$