Math 2401 Quiz 7 Solutions

1. Find the integral $\int_C e^{2x} dr$, where C is the part of the curve $y = e^x$ between $x = 0$ and $x = 3$.

First, for a vector description of C, simply let $t = x$:

$$\mathbf{r}(t) = t\mathbf{i} + e^t\mathbf{j},\ 0 \le t \le 3.$$

Then, $\mathbf{r}'(t) = \mathbf{i} + e^t\mathbf{j}$, and so $|\mathbf{r}'(t)| = \sqrt{1+e^{2t}}$. Hence,

$$\int_C e^{2x} dr = \int_0^3 e^{2t}|\mathbf{r}'(t)|dt = \int_0^3 e^{2t}\sqrt{1+e^{2t}}\,dt = \frac{2(1+e^{2t})^{3/2}}{3(2)}\Bigg|_0^3$$
$$= \frac{1}{3}[(1+e^6)^{3/2} - 2^{3/2}]$$

2. Let S be the surface described by

$$\mathbf{r}(s,t) = s\mathbf{i} + e^s\cos t\mathbf{j} + e^s\sin t\mathbf{k},\ 0 \le s \le 2 \text{ and } 0 \le t \le 2\pi$$

Give an iterated integral for the area of S.

Area of $S = \iint_S d\sigma = \iint_D \left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right| dA$, where D is the recatngle $0 \le s \le 2$, $0 \le t \le 2\pi$.

Now,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^s\cos t & e^s\sin t \\ 0 & -e^s\sin t & e^s\cos t \end{vmatrix} = e^{2s}\mathbf{i} - e^s\cos t\mathbf{j} - e^s\sin t\mathbf{k}.$$

Thus,

$$\left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right| = \sqrt{e^{4s} + e^{2s}\cos^2 t + e^{2s}\sin^2 t} = \sqrt{e^{4s}+e^{2s}}.$$

Finally,

$$\text{Area} = \iint_S d\sigma = \iint_D \left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right| dA$$
$$= \int_0^{2\pi}\int_0^2 \sqrt{e^{4s}+e^{2s}}\,dsdt = \int_0^{2\pi}\int_0^2 e^s\sqrt{e^{2s}+1}\,dsdt$$

3. Give an iterated integral for the volume of the solid bounded by the surface S of the previous problem and the planes $x = 0$ and $x = 2$.

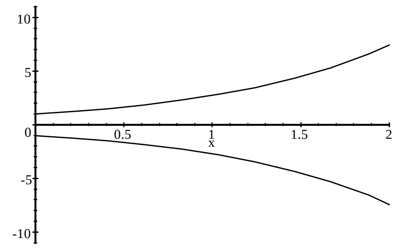
Let's see what this solid looks like. For the surface S, we have

$$x = s,\ y = e^s\cos t, \text{ and } z = e^s\sin t.$$

Thus,

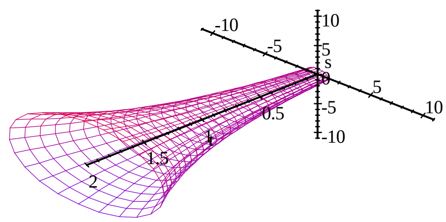
$$y^2 + z^2 = e^{2s}\cos^2 t + e^{2s}\sin^2 t = e^{2s} = e^{2x}$$

Or, $e^{2x} = y^2 + z^2$. When we slice with the plane $z = 0$, we see $e^{2x} = y^2$, or $y = \pm e^x$:

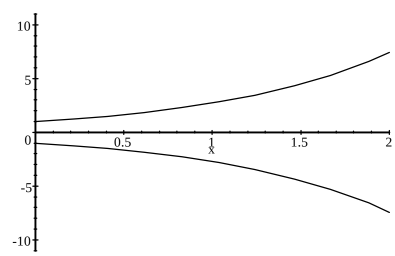


Similarly, if we slice with the plane $z = 0$, we see the same thing: $y = \pm e^x$. Next, if we slice with planes x =constant $= c$, we see circles: $y^2 + z^2 = e^c$.

The surface S thus looks something like a trombone bell:



Let's project onto the $x - y$ plane and thus integrate first with respect to z. We already have the projection; $z = 0$ gives us $y = \pm e^x$:



So, we have

$$V = \iiint_B dV = \iint_P \left(\int_{z=-\sqrt{e^{2x}-y^2}}^{z=\sqrt{e^{2x}-y^2}} dz \right) dA$$

$$= \int_0^2 \int_{-e^x}^{e^x} \int_{-\sqrt{e^{2x}-y^2}}^{\sqrt{e^{2x}-y^2}} dzdydx$$

[There are, of course, other correct possibilities.]

4. Find a point on the surface $\mathbf{r}(s,t) = s^2\mathbf{i} + (s+t)\mathbf{j} + t^2\mathbf{k}$ at which the vector $3\mathbf{i} - 12\mathbf{j} + 2\mathbf{k}$ is normal to the surface, or explain why there is no such point.

A normal to the surface at $\mathbf{r}(s,t)$ is

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2s & 1 & 0 \\ 0 & 1 & 2t \end{vmatrix}$$

$$= 2t\mathbf{i} - 4st\mathbf{j} + 2s\mathbf{k}$$

Therefore, if $3\mathbf{i} - 12\mathbf{j} + 2\mathbf{k}$ is to be normal at $\mathbf{r}(s,t)$, it must be true that

$$\mathbf{N} = 2t\mathbf{i} - 4st\mathbf{j} + 2s\mathbf{k} = \alpha(3\mathbf{i} - 12\mathbf{j} + 2\mathbf{k})$$

for some nonzero scalar α. This gives us three scalar equations:

$$2t = 3\alpha$$

$$-4st = -12\alpha$$

$$2s = 2\alpha$$

We'll simply solve for (s,t). The first and third equations tell us that $t = 3\alpha/2$ and $s = \alpha$. Substituting these into the second equation gives us

$$-4\alpha\frac{3\alpha}{2} = -12\alpha$$

or,

$$\alpha^2 - 2\alpha = \alpha(\alpha - 2) = 0.$$

Now $\alpha = 0$ won't do, so we have $\alpha = 2$. Thus $s = 2$ and $t = 3$, and the point we seek is

$$\mathbf{r}(2,3) = 4\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$$