Chapter Ten

Sequences, Series, and All That

10.1 Introduction

Suppose we want to compute an approximation of the number e by using the Taylor polynomial p_n for $f(x) = e^x$ at $a = 0$. This polynomial is easily seen to be

$$p_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!}.$$

We could now use $p_n(1)$ as an approximation to e. We know from the previous chapter that the error is given by

$$e - p_n(1) = \frac{e}{(n+1)!} 1^{n+1},$$

where $0 < \quad < 1$. Assume we know that $e < 3$, and we have the estimate

$$0 \quad e - p_n(1) \quad \frac{3}{(n+1)!}.$$

Meditate on this error estimate. It tells us that we can make this error as small as we like by choosing n sufficiently large. This is expressed formally by saying that the *limit* of $p_n(1)$ as n *becomes infinite* is e. This is the idea we shall study in this chapter.

10.2 Sequences

A *sequence* of real numbers is simply a function from a subset of the nonnegative integers into the reals. If the domain is infinite, we say the sequence is an ***infinite sequence.*** (Guess what a *finite* sequence is.) We shall be concerned only with infinite sequences, and so the modifier will usually be omitted. We shall also almost always consider sequences in which the domain is either the entire set of nonnegative or positive integers.

There are several notational conventions involved in writing and talking about sequences. If $f: Z_+ \quad \boldsymbol{R}$, it is customary to denote $f(n)$ by f_n, and the sequence itself by (f_n). (Here Z_+ denotes the positive integers.) Thus, for example, $\frac{1}{n}$ is the sequence

f defined by $f(n) = \dfrac{1}{n}$.   The function values f_n are called ***terms*** of the sequence. Frequently one sees a sequence described by writing something like

$$1,4,9,\ldots,n^2,\ldots\ .$$

This is simply another way of describing the sequence (n^2).

Let (a_n) be a sequence and suppose there is a number L such that for any >0, there is an integer N such that $|a_n - L| < $ for all $n > N$. Then L is said to be a ***limit*** of the sequence, and (a_n) is said to ***converge*** to L. This is usually written $\lim\limits_{n} a_n = L$. Now, what does this really mean? It says simply that as n gets big, the terms of the sequence get close to L. I hope it is clear that 0 is a limit of the sequence $\dfrac{1}{n}$.   From the discussion in the Introduction to this chapter, it should be reasonably clear that a limit of the sequence

$$1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n!} \quad \text{is } e\ .$$

The graph of a sequence is pretty dreary compared with the graph of a function whose domain is an interval of reals, but nevertheless, a look at some pictures can help understand some of these definitions. Suppose the sequence (a_n) converges to L. Look at the graph of (a_n):
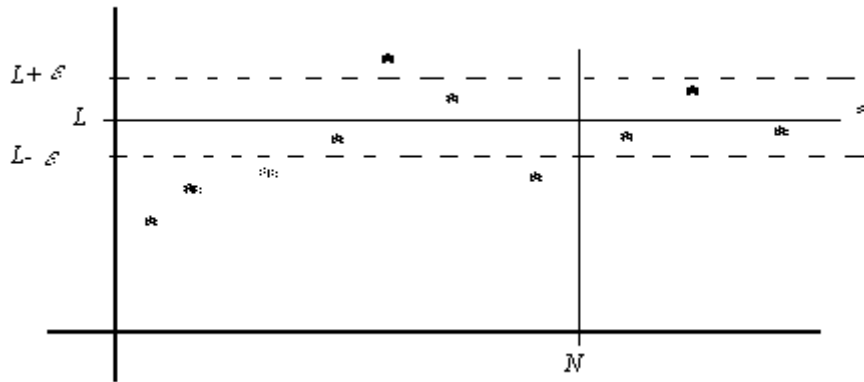
The fact that L is a limit of the sequence means that for any >0, there is an N so that to the right of N, all the spots are in the strip of width 2 centered at L.

Exercises

1. Prove that a sequence can have at most one limit (We may thus speak of *the* limit of a sequence.).

2. Give an example of a sequence that does not have a limit. Explain.

3. Suppose the sequence $(a_n) = a_0, a_1, a_2, \ldots$ converges to L. Explain how you know that the sequence $(a_{n+5}) = a_5, a_6, a_7, \ldots$ also converges to L.

4. Find the limit of the sequence $\dfrac{3}{n^2}$, or explain why it does not converge.

5. Find the limit of the sequence $\dfrac{3n^2 + 2n - 7}{n^2}$, or explain why it does not converge.

6. Find the limit of the sequence $\dfrac{5n^3 - n^2 + 7n + 2}{3n^3 + n^2 - n + 10}$, or explain why it does not converge.

7. Find the limit of the sequence $\dfrac{\log n}{n}$, or explain why it does not converge.

10.3 Series

Suppose (a_n) is a sequence. The sequence $(a_0 + a_1 + \ldots + a_n)$ is called a *series*. It is a little neater to write if we use the usual summation notation: $\displaystyle\sum_{k=0}^{n} a_k$. We have seen an example of such a thing previously; *viz.,*

$$1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!}.$$

It is usual to replace $\displaystyle\lim_{n}\sum_{k=0}^{n} a_k$ by $\displaystyle\sum_{k=0} a_k$. Thus, one would, for example, write

$$e = \sum_{k=0} \frac{1}{k!}.$$

10.3

One also frequently sees the limit $\sum_{k=0}^{\infty} a_k$ written as $a_0 + a_1 + \ldots + a_n + \ldots$. And one more word of warning. Some poor misguided souls also use $\sum_{k=0}^{\infty} a_k$ to stand simply for the series $\sum_{k=0}^{n} a_k$. It is usually clear whether the series or the limit of the series is meant, but it is nevertheless an offensive practice that should be ruthlessly and brutally suppressed.

Example

Let's consider the series $\sum_{k=0}^{n} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n}$. Let

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n}. \text{ Then}$$

$$\frac{1}{2} S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \frac{1}{2^{n+1}}.$$

Thus

$$\frac{S_n}{2} = S_n - \frac{1}{2} S_n = 1 - \frac{1}{2^{n+1}}.$$

This makes it quite easy to see that $\lim_{n} \frac{S_n}{2} = 1$, or $\lim_{n} S_n = 2$. In other words,

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Observe that for series $\sum_{k=0}^{n} a_k$ to converge, it must be true that $\lim_{n} a_n = 0$. To see this, suppose $L = \sum_{k=0}^{\infty} a_k$, and observe that $a_n = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k$. Thus,

$$\lim_{n} a_n = \lim_{n} \left[\sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k \right] = \lim_{n} \sum_{k=0}^{n} a_k - \lim_{n} \sum_{k=0}^{n-1} a_k$$

$$= L - L = 0.$$

In other words, if $\lim_{n} a_n \neq 0$, then the series $\sum_{k=0}^{n} a_k$ does not have a limit.

Another Example

Consider the series $\sum_{k=1}^{n} \dfrac{1}{k}$. First, note that $\lim_{n} \dfrac{1}{k} = 0$. Thus we do not know that the series does not converge; that is, we still don't know anything. Look at the following picture:
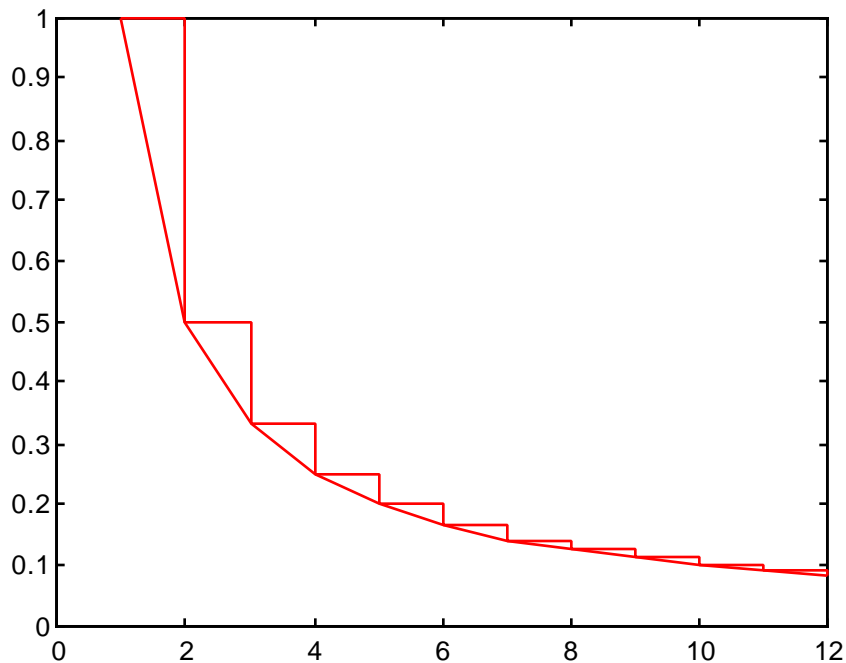
The curve is the graph of $y = \dfrac{1}{x}$. Observe that the area under the "stairs" is simply $\sum_{k=1}^{11} \dfrac{1}{k}$.

Now convince yourself that $\sum_{k=1}^{n} \dfrac{1}{k}$ is larger than the area under the curve $y = \dfrac{1}{x}$ from $x = 1$ to $x = n+1$. In other words,

$$\sum_{k=1}^{n} \dfrac{1}{k} > \int_{1}^{n+1} \dfrac{1}{x}\, dx = \log(n + 1) \,.$$

We know that $\log(n + 1)$ can be made as large as we wish by choosing n sufficiently large.

Thus $\sum_{k=1}^{n} \frac{1}{k}$ can be made as large as we wish by choosing n sufficiently large. From this it

follows that the series $\sum_{k=1}^{n} \frac{1}{k}$ does *not* have a limit. (This series has a name. It is called

the *harmonic series*.)

   The method we used to show that the harmonic series does not converge can be used
on many other series. We simply consider a picture like the one above. Suppose we have a

series $\sum_{k=1}^{n} a_k$ such that $a_k > 0$ for all k . Suppose f is a decreasing function such that

$f(k) = a_k$ for all k . Then if the limit $\lim_{R} \int_{1}^{R} f(x)dx$ does not exist, the series is divergent.

Exercises

8.   Find the limit of the series $\sum_{k=0}^{n} \frac{1}{3^n}$, or explain why it does not converge.

9.   Find the limit of the series $\sum_{k=0}^{n} \frac{5}{\sqrt{n + 3}}$, or explain why it does not converge.

10. Find a value of n that will insure that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > 10^6$.

11. Let $0 \le \theta \le 1$. Prove that $\sin \theta = \sum_{k=0}^{\infty} (-1)^k \frac{\theta^{2k+1}}{(2k + 1)!}$.

   [Hint: $p_{2n+1}(\theta) = \sum_{k=0}^{n} (-1)^k \frac{\theta^{2k+1}}{(2k + 1)!}$ is the Taylor polynomial of degree $\le 2n+1$ for

   the function $f(\theta) = \sin \theta$ at $a = 0$.]

12. Suppose we have a series $\sum\limits_{k=1}^{n} a_k$ such that $a_k > 0$ for all k, and suppose f is a

decreasing function such that $f(k) = a_k$ for all k. Show that if the limit

$$\lim_{R} \int_{1}^{R} f(x)dx$$ exists, then the series is convergent.

13. a) Find all p for which the series $\sum\limits_{k=1}^{n} \dfrac{1}{k^p}$ converges.

b) Find all p for which the series in a) diverges.

10.4 More Series

Consider a series $\sum\limits_{k=0}^{n} a_k$ in which $a_k \geq 0$ for all k. This is called a ***positive***

series. Let $\sum\limits_{k=0}^{n} b_k$ be another positive series. Suppose that $b_k \leq a_k$ for all $k \geq N$, where

N is simply some integer. Now suppose further that we know that $\sum\limits_{k=0}^{n} a_k$ converges.

This tells us all about the series $\sum\limits_{k=0}^{n} b_k$. Specifically, it tells us that this series also

converges. Let's see why that is. First note the obvious: $\sum\limits_{k=0}^{n} b_k$ converges if and only if

$\sum\limits_{k=N}^{n} b_k$ converges. Next, observe that for all n, we have $\sum\limits_{k=N}^{n} b_k \leq \sum\limits_{k=N}^{n} a_k$, from which it

follows at once that $\lim\limits_{n} \sum\limits_{k=N}^{n} b_k$ exists.

Example

What about the convergence of the series $\sum_{k=1}^{n} \dfrac{1}{n^3 + 3n^2 + n + 4}$? Observe first that

$$\dfrac{1}{n^3 + 3n^2 + n + 4} < \dfrac{1}{n^3} .$$ Then observe that the series $\sum_{k=1}^{n} \dfrac{1}{n^3}$ converges because

$\lim\limits_{R} \int_{1}^{R} \dfrac{1}{x^3} dx = \lim\limits_{R} \dfrac{-1}{3R^2} + \dfrac{1}{3} = \dfrac{1}{3}$. Thus $\sum_{k=1}^{n} \dfrac{1}{n^3 + 3n^2 + n + 4}$ converges.

Suppose that, as before, $\sum_{k=0}^{n} a_k$ and $\sum_{k=0}^{n} b_k$ are positive series, and $b_k \le a_k$ for all

$k \ge N$, where N is some number. This time, suppose we know that $\sum_{k=0}^{n} b_k$ is divergent.

Then it should not be too hard for you to convince yourself that $\sum_{k=0}^{n} a_k$ must be

divergent, also.

Exercises

Which of the following series are convergent and which are divergent? Explain your answers.

14. $\sum_{k=0}^{n} \dfrac{1}{2e^k + k}$

15. $\sum_{k=0}^{n} \dfrac{1}{2k + 1}$

16. $\sum_{k=2}^{n} \dfrac{1}{\log k}$

17. $\sum_{k=0}^{n} \dfrac{1}{k^2 + k - 1}$

10.5 Even More Series

We look at one more very nice way to help us determine if a positive series has a limit. Consider a series $\sum_{k=0}^{n} a_k$, and suppose $a_k > 0$ for all k. Next suppose the sequence $\dfrac{a_{k+1}}{a_k}$ is convergent, and let

$$r = \lim_{k} \frac{a_{k+1}}{a_k}.$$

The number r tells us almost everything about the convergence of the series $\sum_{k=0}^{n} a_k$. Let's see about it.

First, suppose that $r < 1$. Then the number $= r + \dfrac{1-r}{2}$ is positive and less than 1. For all sufficiently large k, we know that $\dfrac{a_{k+1}}{a_k}$. In other words, there is an N so that $a_{k+1} \quad a_k$ for all $k \quad N$. Thus

$$a_{k+1} \quad a_k \quad a_{k-1}{}^{2} \quad a_{k-2}{}^{3} \quad \ldots \quad a_N{}^{k+1-N}.$$

Look now at the series

$$\sum_{k=N}^{n} a_N{}^{k-N} = \left(a_N(1 + \quad + \quad{}^{2} + \ldots \quad{}^{n-N}) \right).$$

This one converges because the Geometric series $\sum_{k=0}^{n} {}^{k}$ converges (Recall that $0 < \quad < 1.$). It now follows from the previous section that our original series $\sum_{k=0}^{n} a_k$ has a limit.

A similar argument should convince you that if $r > 1$, then the series $\sum_{k=0}^{n} a_k$ does not have a limit.

The "method" of the previous section is usually called the ***Comparison Test***, while that of this section is usually called the ***Ratio Test***.

10.9

Exercises

Which of the following series are convergent and which are divergent? Explain your answers.

18. $\displaystyle\sum_{k=0}^{n} \frac{10^k}{k!}$

19. $\displaystyle\sum_{k=0}^{n} \frac{3^{2k+1}}{5^k}$

20. $\displaystyle\sum_{k=0}^{n} \frac{3^{2k+1}}{10^k}$

21. $\displaystyle\sum_{k=1}^{n} \frac{3^k}{5^k (k^4 + k + 1)}$

22. $\displaystyle\sum_{k=1}^{n} \frac{3^k (k^4 + k + 1)}{5^k}$

10.6 A Final Remark

The "tests" for convergence of series that we have seen so far all depended on the series having positive terms. We need to say a word about the situations in which this is not necessarily the case. First, if the terms of a series $\displaystyle\sum_{k=0}^{n} a_k$ alternate in sign, and if it is true that $|a_{k+1}| \le |a_k|$ for all k , then $\displaystyle\lim_{k} a_k = 0$ is sufficient to insure convergence of the series. This is not too hard to see—meditate on it for a while.

The second result is a bit harder to see, and we'll just put out the result as the word, asking that you accept it on faith. It says simply that if the series $\displaystyle\sum_{k=0}^{n} |a_k|$ converges, then so also does the series $\displaystyle\sum_{k=0}^{n} a_k$. Thus, faced with an arbitrary series $\displaystyle\sum_{k=0}^{n} a_k$, we

may unleash out arsenal of tests on the series $\sum_{k=0}^{n} |a_k|$. If we find this one to be

convergent, then the original series is also convergent. If, of course, this series turns out not to be convergent, then we still do not know about the original series.