Chapter Eleven

Taylor Series

11.1 Power Series

Now that we are knowledgeable about series, we can return to the problem of investigating the approximation of functions by Taylor polynomials of higher and higher degree. We begin with the idea of a so-called power series. A ***power series*** is a series of the form

$$\left(\sum_{k=0}^{n} c_k (x-a)^k\right).$$

A power series is thus a sequence of special polynomials: each term is obtained from the previous one by adding a constant multiple of the next higher power of $(x-a)$. Clearly the question of convergence will depend on x, as will the limit where there is one. The k th term of the series is $c_k(x-a)^k$ so the Ratio Test calculation looks like

$$r(x) = \lim_{k\to\infty}\left|\frac{c_{k+1}(x-a)^{k+1}}{c_k(x-a)^k}\right| = |x-a| \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|.$$

Recall that our series converges for $r(x) < 1$ and diverges for $r(x) > 1$. Thus this series converges absolutely for all values of x if the number $\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| = 0$. Otherwise, we have absolute convergence for $|x-a| < \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$ and divergence for $|x-a| > \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$. The number $R = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$ is called the ***radius of convergence,*** and the interval $|x-a| < R$ is called the ***interval of convergence.*** There are thus exactly three possibilities for the convergence of our power series $\left(\sum_{k=0}^{n} c_k (x-a)^k\right)$:

(i)The series converges for no value of x except $x = a$; or (ii)The series converges for all values of x; or

(iii)There is a positive number R so that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Note that the Ratio Test tells us nothing about the convergence or divergence of the series at the two points where $|x-a| = R$.

Example

Consider the series $\left(\sum_{k=0}^{n} k!x^k\right)$. Then $R = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = \lim_{k\to\infty}\frac{k!}{(k+1)!} = \lim_{k\to\infty}\frac{1}{k+1} = 0$.

Thus this series converges only when $x = 0$.

Another Example

Now look at the series $\left(\sum_{k=0}^{n} 3^k (x-1)^k\right)$. Here $R = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right| = \lim_{k\to\infty}\frac{3^k}{3^{k+1}} = \lim_{k\to\infty}\frac{1}{3} = \frac{1}{3}$.

Thus, this one converges for $|x-1| < \frac{1}{3}$ and diverges for $|x-1| > \frac{1}{3}$.

Exercises

Find the interval of convergence for each of the following power series:

1. $\left(\sum_{k=0}^{n} (x+5)^k\right)$

2. $\left(\sum_{k=0}^{n} \frac{1}{k}(x-1)^k\right)$

3. $\left(\sum_{k=0}^{n} \frac{k}{3k+1}(x-4)^k\right)$

4. $\left(\sum_{k=0}^{n} \frac{3^k}{k!}(x+1)^k\right)$

5. $\left(\sum_{k=0}^{n} \frac{k!}{7(k^2+1)}(x-9)^k\right)$

11.2 Limit of a Power Series

If the interval of convergence of the power series $\left(\sum_{k=0}^{n} c_k(x-a)^k\right)$ is $|x-a|<R$, then, of course, the limit of the series defines a function f:

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k, \text{ for } |x-a|<R.$$

It is known that this function has a derivative, and this derivative is the limit of the derivative of the series. Moreover, the differentiated series has the same interval of convergence as that of the series defining f. Thus for all x in the interval of convergence, we have

$$f'(x) = \sum_{k=1}^{\infty} kc_k(x-a)^{k-1}.$$

We can now apply this result to the power series for the derivative and conclude that f has all derivatives, and they are given by

$$f^{(p)}(x) = \sum_{k=p}^{\infty} k(k-1)\ldots(k-p+1)c_k(x-a)^{k-p}.$$

Example

We know that $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$. It follows that

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

for $|x| < 1$.

It is, miraculously enough, also true that the limit of a power series can be integrated, and the integral of the limit is the limit of the integral. Once again, the interval of convergence of the integrated series remains the same as that of the original series:

$$\int_a^x f(t)dt = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(x-a)^{k+1}.$$

Example

We may simply integrate the Geometric series to get

$$\log(1-x) = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}, \text{ for } -1 < x < 1, \text{ or } 0 < 1-x < 2.$$

It is also valid to perform all the usual arithmetic operations on power series. Thus if $f(x) = \sum_{k=0}^{\infty} c_k x^k$ and $g(x) = \sum_{k=0}^{\infty} d_k x^k$ for $|x| < r$, then

$$f(x) \pm g(x) = \sum_{k=0}^{\infty} (c_k \pm d_k) x^k, \text{ for } |x| < r.$$

Also,

$$f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} c_i d_{k-i} \right) c_k x^k, \text{ for } |x| < r.$$

The essence of the story is that power series behave as if they were “infinite degree” polynomials—the limits of power series are just about the nicest functions in the world.

Exercises

6. What is the limit of the series $\left(\sum_{k=0}^{n} x^{2k}\right)$? What is its interval of convergence?

7. What is the limit of the series $\left(\sum_{k=1}^{n} 2(-1)^k kx^{2k-1}\right)$? What is its interval of convergence?

8. Find a power series that converges to $\tan^{-1} x$ on some nontrivial interval.

9. Suppose $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$. What is $f^{(p)}(a)$?

11.3 Taylor Series

Our major interest in finding a power series that converges to a given function. The obvious candidate for such a series is simply the sequence of Taylor polynomials of increasing degree. Thus if f is a given function, and a is a point in the interior of the domain of f, the ***Taylor Series for f** at a* is the series

$$\left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right).$$

The Taylor Series is thus an "infinite degree" Taylor Polynomial>

In general, the Taylor series for a function may not converge on any nontrivial interval to f, but, mercifully, for many sufficiently nice functions it does. In such cases, we are provided with the nice answer to the question proposed back in Chapter Nine: Can we approximate the function f as well as we like by a Taylor Polynomial for sufficiently large degree?

Example

The Taylor series for $f(x) = \sin x$ at $x = a$ is simply $\left(\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)$. An easy calculation shows us that the radius of convergence is infinite, or in other words, this power series converges for all x. But is the limit $\sin x$? That's easy to decide. From Section 9.3, we know that

$$\left|\sin x - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right| \le \frac{|x|^{2n+3}}{(2n+3)!},$$

and we know that

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0,$$

no mater what x is. Thus we have

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \text{ for all } x.$$

Exercises

10. Find the Taylor Series at $a = 0$ for $f(x) = e^x$. Find the interval of convergence and show that the series converges to f on this interval.

11. Find the Taylor Series at $a = 0$ for $f(x) = \cos x$. Find the interval of convergence and show that the series converges to f on this interval.

12. Find the derivative of the cosine function by differentiating the Taylor Series you found in Problem #11.

13. Find the Taylor Series at $a = 1$ for $f(x) = \log x$. Find the interval of convergence and show that the series converges to f on this interval.

14. Let the function f be defined by

$$f(x) = \begin{cases} 0, \text{ for } x = 0 \\ e^{-1/x^2}, \text{ for } x \neq 0 \end{cases}.$$

Find the Taylor Series at $a = 0$ for f. Find the interval of convergence and the limit of the series.