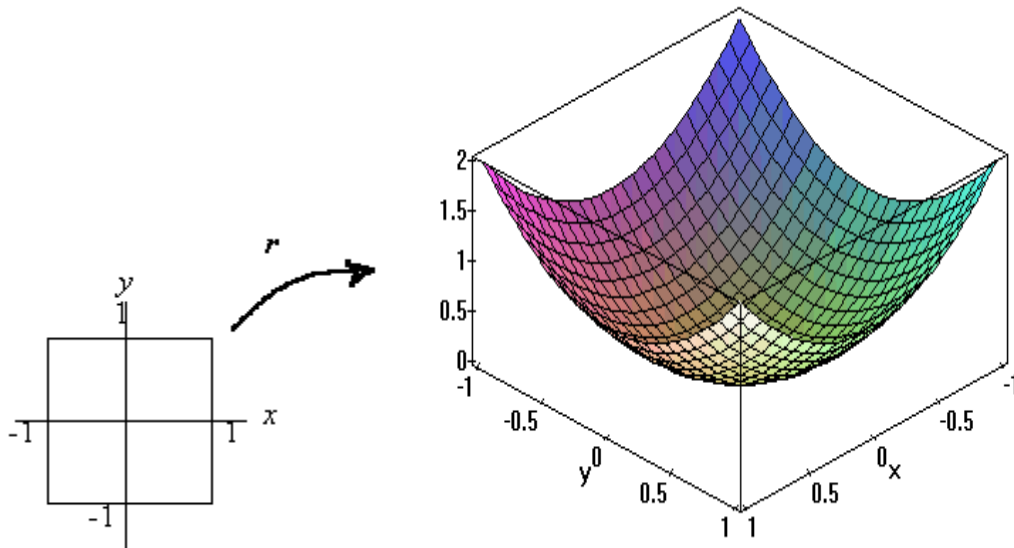


## Chapter Fifteen

# Surfaces Revisited

### 15.1 Vector Description of Surfaces

We look now at the very special case of functions  $r: D \rightarrow \mathbb{R}^3$ , where  $D \subset \mathbb{R}^2$  is a nice subset of the plane. We suppose  $r$  is a nice function. As the point  $(s, t) \in D$  moves around in  $D$ , if we place the tail of the vector  $r(s, t)$  at the origin, the nose of this vector will trace out a surface in three-space. Look, for example at the function  $r: D \rightarrow \mathbb{R}^3$ , where  $r(s, t) = si + tj + (s^2 + t^2)k$ , and  $D = \{(s, t) \in \mathbb{R}^2 : -1 \leq s, t \leq 1\}$ . It shouldn't be difficult to convince yourself that if the tail of  $r(s, t)$  is at the origin, then the nose will be on the paraboloid  $z = x^2 + y^2$ , and for all  $(s, t) \in D$ , we get the part of the paraboloid above the square  $-1 \leq x, y \leq 1$ . It is sometimes helpful to think of the function  $r$  as providing a map from the region  $D$  to the surface.



The vector function  $r$  is called a *vector description* of the surface. This is, of course, exactly the two dimensional analogue of the vector description of a curve.

For a curve,  $\mathbf{r}$  is a function from a nice piece of the real line into three space; and for a surface,  $\mathbf{r}$  is a function from a nice piece of the plane into three space.

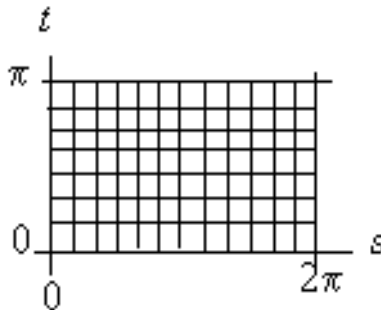
Let's look at another example. Here, let

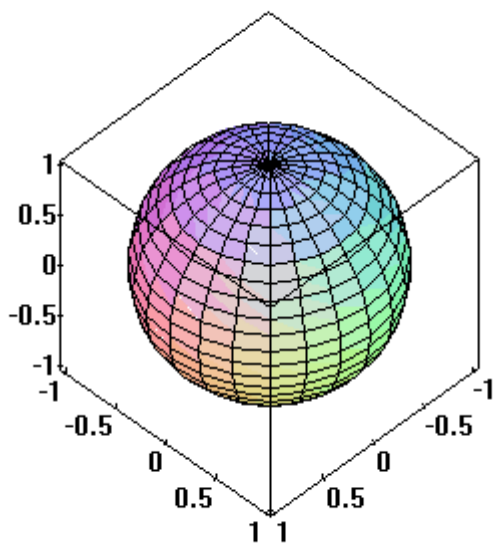
$$\mathbf{r}(s, t) = \cos s \sin t \mathbf{i} + \sin s \sin t \mathbf{j} + \cos t \mathbf{k} ,$$

for  $0 \leq t \leq \pi$  and  $0 \leq s \leq 2\pi$ . What have we here? First, notice that

$$\begin{aligned} |\mathbf{r}(s, t)|^2 &= (\cos s \sin t)^2 + (\sin s \sin t)^2 + (\cos t)^2 \\ &= \sin^2 t (\cos^2 s + \sin^2 s) + \cos^2 t \\ &= \sin^2 t + \cos^2 t = 1 \end{aligned}$$

Thus the nose of  $\mathbf{r}$  is always on the sphere of radius one and centered at the origin. Notice next, that the variable, or parameter,  $s$  is the longitude of  $\mathbf{r}(s, t)$ ; and the variable  $t$  is the latitude of  $\mathbf{r}(s, t)$ . (More precisely,  $t$  is co-latitude.) A moment's reflection on this will convince you that as  $\mathbf{r}$  is a description of the entire sphere. We have a map of the sphere on the rectangle



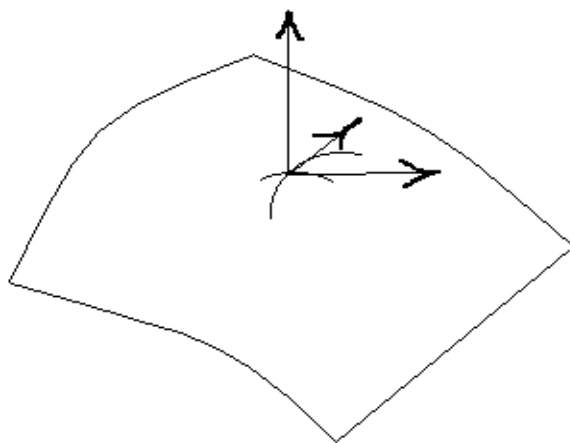


Observe that the entire lower edge of the rectangle (the line from  $(0,0)$  to  $(2,0)$ ) is mapped by  $\mathbf{r}$  onto the North Pole, while the upper edge is mapped onto the South Pole.

Let  $\mathbf{r}(s, t)$ ,  $(s, t) \in D$  be a vector description of a surface  $S$ , and let  $\mathbf{p} = \mathbf{r}(\bar{s}, \bar{t})$  be a point on  $S$ . Now,  $\mathbf{c}(s) = \mathbf{r}(s, \bar{t})$  is a curve on the surface that passes through the point  $\mathbf{p}$ .

Thus the vector  $\frac{d\mathbf{c}}{ds} = \frac{\mathbf{r}}{s}(\bar{s}, \bar{t})$  is tangent to this curve at the point  $\mathbf{p}$ . We see in the same

way that the vector  $\frac{\mathbf{r}}{t}(\bar{s}, \bar{t})$  is tangent to the curve  $\mathbf{r}(\bar{s}, t)$  at  $\mathbf{p}$ .



At the point  $p = r(\bar{s}, \bar{t})$  on the surface  $S$ , the vectors  $\frac{\mathbf{r}}{s}$  and  $\frac{\mathbf{r}}{t}$  are thus tangent to  $S$ .

Hence the vector  $\frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t}$  is normal to  $S$ .

### Example

Let's find a vector normal to the surface given by the vector description  $\mathbf{r}(s, t) = s\mathbf{i} + t\mathbf{j} + (s^2 + t^2)\mathbf{k}$  at a point. We need to find the partial derivatives  $\frac{\mathbf{r}}{s}$  and

$\frac{\mathbf{r}}{t}$ :

$$\frac{\mathbf{r}}{s} = \mathbf{i} + 2s\mathbf{k}, \text{ and } \frac{\mathbf{r}}{t} = \mathbf{j} + 2t\mathbf{k}.$$

The normal  $N$  is

$$N = \frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2s \\ 0 & 1 & 2t \end{vmatrix} = -2s\mathbf{i} - 2t\mathbf{j} + \mathbf{k}.$$

Meditate on the geometry here and convince yourself that this result is at least reasonable.

### Exercises

1. Give a vector description for the surface  $z = \sqrt{x + 2y^2}$ ,  $x, y \geq 0$ .
2. Give a vector description for the ellipsoid  $4x^2 + y^2 + 8z^2 = 16$ .
3. Give a vector description for the cylinder  $x^2 + y^2 = 1$ .

4. Describe the surface given by  $\mathbf{r}(s, t) = s \cos t \mathbf{i} + s \sin t \mathbf{j} + s \mathbf{k}$ ,  $0 \leq t < 2\pi$ ,  $-1 \leq s \leq 1$ .
5. Describe the surface given by  $\mathbf{r}(s, t) = s \cos t \mathbf{i} + s \sin t \mathbf{j} + s^2 \mathbf{k}$ ,  $0 \leq t < 2\pi$ ,  $1 \leq s \leq 2$ .
6. Give a vector description for the sphere having radius 3 and centered at the point (1,2,3).
7. Find an equation (I.e., a vector description) of the line normal to the sphere  $x^2 + y^2 + z^2 = a^2$  at the point  $(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}})$ .
8. Find a scalar equation (I.e., of the form  $f(x, y, z) = 0$ ) of the plane tangent to the sphere  $x^2 + y^2 + z^2 = a^2$  at the point  $(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}})$ .
9. Find all points on the surface  $\mathbf{r}(s, t) = (s^2 + t^2)\mathbf{i} + (s + 3t)\mathbf{j} - st\mathbf{k}$  at which the tangent plane is parallel to the plane  $5x - 6y + 2z = 7$ , or show there are no such points.
10. Find an equation of the plane that contains the point (1,-2,3) and is parallel to the plane tangent to the surface  $\mathbf{r}(s, t) = (s + t)\mathbf{i} + s^2\mathbf{j} - 2t^2\mathbf{k}$  at the point (1, 4,-18).

## 15.2 Integration

Suppose we have a nice surface  $S$  and a function  $f: S \rightarrow \mathbf{R}$  defined on the surface. We want to define an integral of  $f$  on  $S$  as the limit of some sort of Riemann sum in the way in which we have already defined various integrals. Here we have a slight problem in that we really are not sure at this point exactly what we might mean by the *area* of a

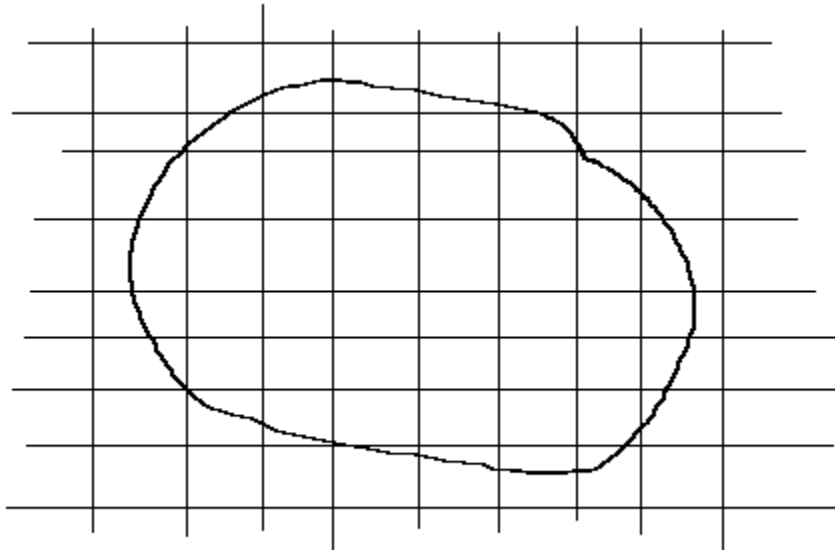
small piece of surface. We assume the surface is sufficiently smooth to allow us to approximate the area of a small piece of it by a small planar region, and then add up these approximations to get a Riemann sum, *etc.*, *etc.* Let's be specific.

We subdivide  $S$  into a number of small pieces  $S_1, S_2, \dots, S_n$  each having area  $A_i$ , select points  $\mathbf{r}_i^* = (x_i^*, y_i^*, z_i^*) \in S_i$ , and form the Riemann sum

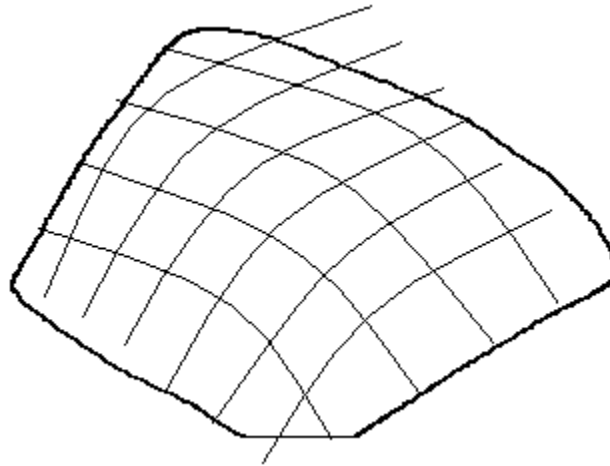
$$R = \sum_{i=1}^n f(\mathbf{r}_i^*) A_i .$$

Then, of course, we take finer and finer subdivisions, and if the corresponding Riemann sums have a limit, this limit is the thing we call the integral of  $f$  on  $S$ :  $\int_S f(\mathbf{r}) dS$ .

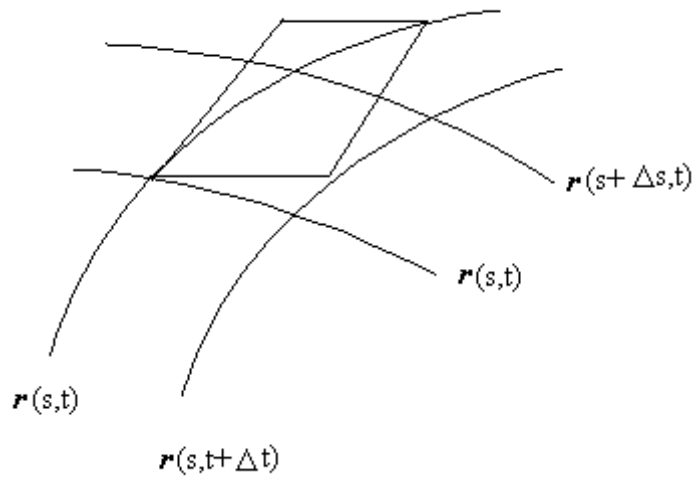
Now, how do we find such a thing. We need a vector description of  $S$ , say  $\mathbf{r}: D \rightarrow \mathbb{R}^3$ ,  $\mathbf{r}(D) = S$ . The surface  $S$  is subdivided by subdividing the region  $D \subset \mathbb{R}^2$  into rectangles in the usual way:



The images of the vertical lines,  $s = \text{constant}$ , form a family of "parallel" curves on the surface, and the images of the horizontal lines  $t = \text{constant}$ , also form a family of such curves:



Let's look closely at one of the subdivisions:



We paste a parallelogram tangent to the surface at the point  $\mathbf{r}(s_i, t_i)$  as shown. The lengths of the sides of this parallelogram are  $\left| \frac{\mathbf{r}}{s}(s_i, t_i) \right| \Delta s_i$  and  $\left| \frac{\mathbf{r}}{t}(s_i, t_i) \right| \Delta t_i$ . The area is then  $\left| \frac{\mathbf{r}}{s}(s_i, t_i) \times \frac{\mathbf{r}}{t}(s_i, t_i) \right| \Delta s_i \Delta t_i$ , and we use the approximation

$$A_i \approx \left| \frac{\mathbf{r}}{s}(s_i, t_i) \times \frac{\mathbf{r}}{t}(s_i, t_i) \right| \Delta s_i \Delta t_i$$

in the Riemann sums:

$$R = \sum_{i=1}^n f(\mathbf{r}(s_i, t_i)) \left| \frac{\mathbf{r}}{s}(s_i, t_i) \times \frac{\mathbf{r}}{t}(s_i, t_i) \right| \Delta s_i \Delta t_i.$$

These are just the Riemann sums for the usual old time double integral of the function

$$F(s, t) = \sum_{i=1}^n f(\mathbf{r}(s_i, t_i)) \left| \frac{\mathbf{r}}{s}(s_i, t_i) \times \frac{\mathbf{r}}{t}(s_i, t_i) \right|$$

over the plane region  $D$ . Thus,

$$\int_S f(\mathbf{r}) dS = \int_D f(\mathbf{r}(s, t)) \left| \frac{\mathbf{r}}{s}(s, t) \times \frac{\mathbf{r}}{t}(s, t) \right| dA.$$

### Example

Let's use our new-found knowledge to find the area of a sphere of radius  $a$ . Observe that the area of a surface  $S$  is simply the integral  $\int_S dS$ . In the previous section, we found a vector description of the sphere:



$$\mathbf{r}(s, t) = a \cos s \sin t \mathbf{i} + a \sin s \sin t \mathbf{j} + a \cos t \mathbf{k},$$

$0 < t < \pi$  and  $0 < s < 2\pi$ . Compute the partial derivatives:

$$\frac{\mathbf{r}}{s} = -a \sin s \sin t \mathbf{i} + a \cos s \sin t \mathbf{j}, \text{ and}$$

$$\frac{\mathbf{r}}{t} = a \cos s \cos t \mathbf{i} + a \sin s \cos t \mathbf{j} - a \sin t \mathbf{k}$$

Then

$$\begin{aligned} \frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} &= a^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin s \sin t & \cos s \sin t & 0 \\ \cos s \cos t & \sin s \cos t & -\sin t \end{vmatrix} \\ &= a^2 [-\cos s \sin^2 t \mathbf{i} - \sin s \sin^2 t \mathbf{j} - \sin t \cos t \mathbf{k}] \end{aligned}$$

Next we need to find the length of this vector:

$$\begin{aligned} \left| \frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} \right| &= a^2 [\cos^2 s \sin^4 t + \sin^2 s \sin^4 t + \sin^2 t \cos^2 t]^{1/2} \\ &= a^2 [\sin^4 t + \sin^2 t \cos^2 t]^{1/2} = a^2 [\sin^2 t (\sin^2 t + \cos^2 t)]^{1/2} \\ &= a^2 |\sin t| \end{aligned}$$

Hence,

$$\text{Area} = \int_D dS = \int_D \left| \frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} \right| dA = \int_D a^2 |\sin t| dA$$

$$\begin{aligned}
 &= a^2 \int_0^{\pi/2} |\sin t| ds dt \\
 &= 2 a^2 \int_0^{\pi/2} \sin t dt = 4 a^2
 \end{aligned}$$

### Another Example

Let's find the centroid of a hemispherical shell  $H$  of radius  $a$ . Choose our coordinate system so that the shell is the surface  $x^2 + y^2 + z^2 = a^2$ ,  $z \geq 0$ . The centroid  $(\bar{x}, \bar{y}, \bar{z})$  is given by

$$\bar{x} = \frac{\int_H x dS}{\int_H dS} \quad \bar{y} = \frac{\int_H y dS}{\int_H dS} \quad \text{and} \quad \bar{z} = \frac{\int_H z dS}{\int_H dS}.$$

First, note from the symmetry of the shell that  $\bar{x} = \bar{y} = 0$ . Second, it should be clear from the previous example that  $\int_H dS = 2 a^2$ . This leaves us with just one integral to evaluate:

$\int_H z dS$ . Most of the work was done in the example before this one. This hemisphere has the same vector description as the sphere, except for the fact that the domain of  $\mathbf{r}$  is the rectangle  $0 \leq s \leq 2\pi$ ,  $0 \leq t \leq \frac{\pi}{2}$ . Thus

$$\begin{aligned}
 \int_H z dS &= a^2 \int_0^{1/2} \int_0^{1/2} a \cos t \left| \frac{\mathbf{r}}{s} \times \frac{\mathbf{r}}{t} \right| ds dt \\
 &= a^3 \int_0^{1/2} \cos t \sin t ds dt = 2 a^3 \int_0^{1/2} \cos t \sin t dt \\
 &= a^3 \sin^2 t \Big|_0^{1/2} = a^3
 \end{aligned}$$

And so we have  $\bar{z} = \frac{a^3}{2 a^2} = \frac{a}{2}$ . Is this the result you expected?

### Yet One More Example

Our new definition of a surface integral certainly includes the old one for plane surfaces. Look at the "surface" described by the vector function

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j},$$

with  $\mathbf{r}$  defined on some subset  $D$  of the  $r - \theta$  plane. For what we hope will be obvious reasons, we are using the letters  $r$  and  $\theta$  instead of  $s$  and  $t$ . Now consider an integral

$$\int_S f(x, y) dS$$

over the surface  $S$  described by  $\mathbf{r}$ . We know this integral to be given by

$$\int_S f(x, y) dS = \int_D f(r \cos \theta, r \sin \theta) \left| \frac{\mathbf{r}}{r} \times \frac{\mathbf{r}}{r} \right| dA.$$

Let's find the partial derivatives:

$$\frac{\mathbf{r}}{r} = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j}, \text{ and}$$

$$\frac{\mathbf{r}}{r} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}.$$

Thus,

$$\frac{\mathbf{r}}{r} \times \frac{\mathbf{r}}{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r \sin \theta & r \cos \theta & 0 \\ \cos \theta & \sin \theta & 0 \end{vmatrix} = -r \mathbf{k},$$

and we have  $\left| \frac{\mathbf{r}}{r} \times \frac{\mathbf{r}}{r} \right| = r$ . Hence,

$$\int_S f(x, y) dS = \int_D f(r \cos \theta, r \sin \theta) \left| \frac{\mathbf{r}}{r} \times \frac{\mathbf{r}}{r} \right| dA = \int_D f(r \cos \theta, r \sin \theta) r dA.$$

This should look familiar!

### Exercises

**11.** Find the area of that part of the surface  $z = x^2 + y^2$  that lies between the planes  $z = 1$  and  $z = 2$ .

**12.** Find the centroid of the surface given in Problem **11**.

**13.** Find the area of that part of the Earth that lies North of latitude  $45^\circ$ . (Assume the surface of the Earth is a sphere.)

**14.** A spherical shell of radius  $a$  is centered at the origin. Find the centroid of that part of it which is in the first octant.

15. a) Find the centroid of the solid right circular cone having base radius  $a$  and altitude  $h$ .

b) Find the centroid of the lateral surface of the cone in part a).

16. Find the area of the ellipse cut from the plane  $z = 2x$  by the cylinder  $x^2 + y^2 = 1$ .

17. Evaluate  $\int_S (x + y + z) dS$ , where  $S$  is the surface of the cube cut from the first octant

by the planes  $x = a$ ,  $y = a$ , and  $z = a$ .

18. Evaluate  $\int_S x\sqrt{y^2 + 1} dS$ , where  $S$  is the surface cut from the paraboloid

$y^2 + 4z = 16$  by the planes  $x = 0$ ,  $x = 1$ , and  $z = 0$ .