Math 2401 Final Examination Solutions

1. Find the unit tangent, the principal normal, and the curvature of the curve described by

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{k}.$$

First,

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 3\cos t\mathbf{k};$$

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4\sin^2 t + 9\cos^2 t} = \sqrt{4 + 5\cos^2 t}.$$

The unit tangent $\mathbf{T}$ is thus

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{4 + 5\cos^2 t}}(-2\sin t\mathbf{i} + 3\cos t\mathbf{k}).$$

Now,

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$$

so let's find $\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt}$. First,

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= \frac{1}{\sqrt{4 + 5\cos^2 t}}(-2\cos t\mathbf{i} - 3\sin t\mathbf{k}) + \\ &\quad \frac{10\cos t\sin t}{2(4 + 5\cos^2 t)^{3/2}}(-2\sin t\mathbf{i} + 3\cos t\mathbf{k}) \\ &= \frac{1}{(4 + 5\cos^2 t)^{3/2}}\Big[\big(-8\cos t - 10\cos^3 t - 10\cos t\sin^2 t\big)\mathbf{i} + \\ &\quad (-12\sin t - 15\cos^2 t\sin t + 15\cos^2 t\sin t)\mathbf{k}\Big] \\ &= \frac{1}{(4 + 5\cos^2 t)^{3/2}}(-18\cos t\mathbf{i} - 12\sin t\mathbf{k})\end{aligned}$$

Thus,

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt}\Big/\frac{ds}{dt} = \frac{-6}{(4 + 5\cos^2 t)^2}(-3\cos t\mathbf{i} - 2\sin t\mathbf{k}).$$

We have

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{6}{\sqrt{13}\,(4 + 5\cos^2 t)^2}$$

and

$$\mathbf{N} = \frac{1}{\sqrt{13}}(-3\cos t\mathbf{i} - 2\sin t\mathbf{k}).$$

2. Find the directional derivative of

$$f(x,y,z) = z\ln\left(\frac{x}{y}\right)$$

at $(2,2,1)$ in the direction toward the point $(1,1,2)$.

The desired directional derivative is simply $\nabla f(2,2,1) \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction of the vector $\mathbf{v} = (1-2)\mathbf{i} + (1-2)\mathbf{j} + (2-1)\mathbf{k} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Thus $\mathbf{u} = \frac{1}{\sqrt{3}}(-\mathbf{i} - \mathbf{j} + \mathbf{k})$.

Next,

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= \frac{z}{x}\mathbf{i} - \frac{z}{y}\mathbf{j} + \ln\left(\frac{x}{y}\right)\mathbf{k}.\end{aligned}$$

At the point $(2,2,1)$, this becomes

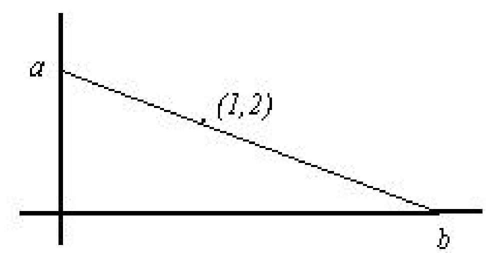
$$\nabla f = \frac{1}{2}(\mathbf{i} - \mathbf{j}),$$

and so the directional derivative $D_{\mathbf{u}}f$ is

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{2}(\mathbf{i} - \mathbf{j}) \cdot \frac{1}{\sqrt{3}}(-\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0.$$

3. Of all lines passing through the point $(1,2)$, find one that maximizes the area of the triangle bounded by the lines and the coordinate axes.

Take a look:

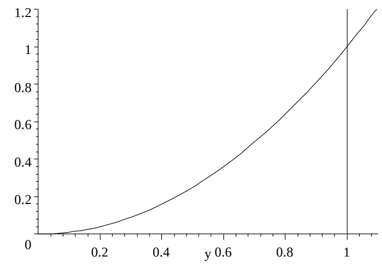


The area of the triangle is $\frac{ab}{2}$ and I hope it is clear we can make this just as large as we wish by taking say a sufficiently large. There is thus **no maximum area**.

4. Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3 + 1)dxdy.$$

I can not find an antiderivative for $\sin(x^3 + 1)$, so let's reconstruct the two-dimensional integral from whence the iterated integral came and try integrating in the other order. We need a picture:



Now then,

$$\int_0^1 \int_{\sqrt{y}}^1 \sin(x^3+1)dxdy = \int_0^1 \int_0^{x^2} \sin(x^3+1)dydx$$

$$= \int_0^1 x^2 \sin(x^3+1)dx$$

$$= -\frac{1}{3}\cos(x^3+1)\Big|_0^1 = \frac{\cos 1 - \cos 2}{3}$$

5. Find the centroid of the curve $y = -x^2, -1 \leq x \leq 1$.

We know the centroid $(\tilde{x}, \tilde{y})$ is given by

$$\tilde{x} = \frac{\int_C xds}{\int_C ds}, \text{ and } \tilde{y} = \frac{\int_C yds}{\int_C ds}.$$

To evaluate the line integrals we need a vector description of the curve C. This is easy; simply take $t = x$:

$$\mathbf{r}(t) = t\mathbf{i} - t^2\mathbf{j}, \ -1 \leq t \leq 1.$$

Then $\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$ and so $|\mathbf{r}'(t)| = \sqrt{1+4t^2}$. Now the scalar line integrals are

$$\int_C xds = \int_{-1}^1 t|\mathbf{r}'(t)|dt = \int_{-1}^1 t\sqrt{1+4t^2}\,dt;$$

$$\int_C yds = \int_{-1}^1 -t^2|\mathbf{r}'(t)|dt = \int_{-1}^1 -t^2\sqrt{1+4t^2}\,dt; \text{ and}$$

$$\int_C ds = \int_{-1}^1 |\mathbf{r}'(t)|dt = \int_{-1}^1 \sqrt{1+4t^2}\,dt.$$

Hence

$$\tilde{x} = \frac{\int_C x\,ds}{\int_C ds} = \frac{\int_{-1}^{1} t\sqrt{1+4t^2}\,dt}{\int_{-1}^{1} \sqrt{1+4t^2}\,dt}$$

$$\tilde{y} = \frac{\int_C y\,ds}{\int_C ds} = \frac{\int_{-1}^{1} -t^2\sqrt{1+4t^2}\,dt}{\int_{-1}^{1} \sqrt{1+4t^2}\,dt}.$$

6. Find the total flux of $x\mathbf{i} + y\mathbf{j}$ out of the solid bounded above by the surface $z = 1$ and below by the surface $z = x^2 + y^2$.

Use the Divergence Theorem (aka Gauss's Theorem):

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_B (\nabla \cdot \mathbf{F})dV.$$

Here, $\nabla \cdot \mathbf{F} = 1 + 1 = 2,$ and so we have

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = 2\iiint_B dV.$$

To evaluate $\iiint_B dV$, project the solid down onto the $x - y$ plane and use polar coordinates:

$$\iiint_B dV = \iint_C \left(\int_{x^2+y^2}^{1} dz \right) dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r\,dr\,d\theta = \frac{\pi}{2}.$$

Thus the total flux is

$$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = 2\iiint_B dV = 2\frac{\pi}{2} = \pi.$$