Math 4581 Final Examination Spring 2001- Solutions

1. In the space of all continuous real-valued functions on the interval $[0,1]$ with the inner product $(f,g) = \int_0^1 f(x)g(x)dx$, let V be the space spanned by $\{1, \sqrt{x}\}$.

a)Find an orthogonal base for V.

Use Gram-Schmidt:

Let $u_1 = 1$. Then

$$u_2 = \sqrt{x} - \frac{(u_1, \sqrt{x})}{(u_1, u_1)} u_1$$

$$= \sqrt{x} - \frac{\int_0^1 \sqrt{x}\, dx}{\int_0^1 dx}$$

:

$$= \sqrt{x} - \frac{2}{3}$$

Thus our orthogonal base is

$$\left\{1, \sqrt{x} - \frac{2}{3}\right\}$$

b)What is the dimension of V? Explain.

The above collection is a base for V. It spans, and being orthogonal, it is independent. It contains two items, so the dimension of V is 2.

c)Find the projection of x onto V..

$$proj(x : u_1, u_2) = \frac{(x, u_1)}{(u_1, u_1)} u_1 + \frac{(x, u_2)}{(u_2, u_2)} u_2.$$

Now,

$$(x, u_1) = \int_0^1 x dx = \frac{1}{2}$$

$$(u_1, u_1) = 1$$

$$(x, u_2) = \int_0^1 x\left(\sqrt{x} - \frac{2}{3}\right) dx = \frac{1}{15}$$

$$(u_2, u_2) = \int_0^1 \left(\sqrt{x} - \frac{2}{3} \right)^2 dx = \frac{1}{18}$$

Finally,

$$\begin{aligned} proj(x : u_1, u_2) &= \frac{1}{2} + \frac{18}{15}\left(\sqrt{x} - \frac{2}{3} \right) \\ &= -\frac{3}{10} + \frac{6}{5}\sqrt{x} \end{aligned}$$

e)Find an element g in V such that

$$\int_0^1 [g(x) - x]^2 dx \leq \int_0^1 [f(x) - e^{2x}]^2 dx$$

for all f in V.

This is, of course, simply the projection of x onto V, and we have just found this to be

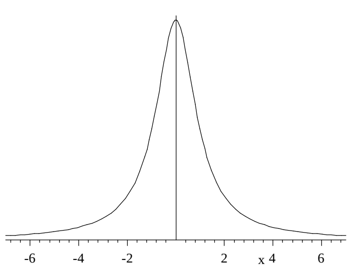
$$proj(x : u_1, u_2) = -\frac{3}{10} + \frac{6}{5}\sqrt{x}$$

2. Let

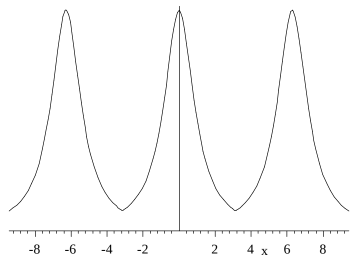
$$f(x) = \frac{1}{x^2 + 1}.$$

Let $c(x)$ be the limit of the Fourier cosine series for f on the interval $[0, \pi]$; let $s(x)$ be the limit of the Fourier sine series for f on $[0, \pi]$; let $C(x)$ be the limit of the Fourier cosine integral for f; and let $S(x)$ be the limit of the Fourier sine integral for f. Sketch the graph of each of the functions $c(x)$, $s(x)$, $C(x)$, and $S(x)$ on the interval $[-3\pi, 3\pi]$.

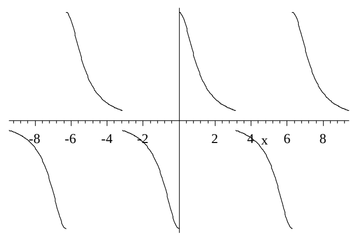
For your viewing pleasure, here is a picture of f:



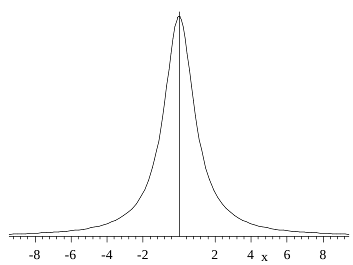
$c(x)$: even periodic extension of f on $[0, \pi]$:



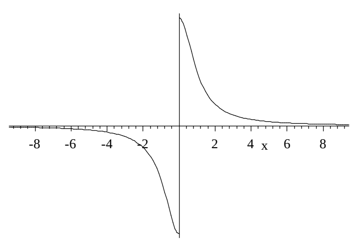
$s(x)$: odd periodic extension of *f* on $[0,\pi]$:



$C(x)$: even extension of *f* (This is simply *f* since *f* is even.)



$S(x)$: odd extension of *f* to entire real line:



3. a)Find all eigenvalues and corresponding eigenfunctions:

$$\varphi''(x) = -\lambda^2\varphi(x)$$
$$\varphi'(0) = \varphi'(\pi) = 0$$

For $\lambda \neq 0$:

$$\varphi(x) = A\cos\lambda x + B\sin\lambda x$$
$$\varphi'(x) = \lambda(-A\sin\lambda x + B\cos\lambda x)$$
$$\varphi'(0) = \lambda B = 0$$

Thus $B = 0$ since $\lambda \neq 0$. Then

$$\varphi'(\pi) = -\lambda A\sin\lambda\pi = 0$$

Thus, as happens so often, we have $\lambda_n = 1,2,3,\ldots,$ with corresponding eigenfunctions $\varphi_n = \cos\lambda_n x = \cos nx$.

Now, what about $\lambda = 0$? In this case, our differential equation becomes $\varphi''(x) = 0$. Thus

$$\varphi(x) = Ax + B, \text{ and}$$
$$\varphi'(x) = A$$

Hence $\varphi'(0) = A = 0$ means $A = 0$. So $\lambda_0 = 0$ is also an eigenvalue with eigenfunction $\varphi_0 = 1$.

4. Solve:

$$u_{xx} - u_{tt} = 0,\ 0 < x < \pi,\ t > 0$$
$$u_x(0,1) = u_x(\pi,t) = 0$$
$$u(x,0) = 5 + \cos 2x, \text{ and } u_t(x,0) = 1.$$

From Problem 3, we know to let

$$u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t)\cos nx.$$

The equation becomes

$$u_{xx} - u_{tt} = -\alpha_n''(t) + \sum_{n=1}^{\infty}[-n^2\alpha_n(t) - \alpha_n''(t)]\cos nx = 0.$$

Then $-n^2\alpha_n(t) - \alpha_n''(t) = 0$ gives us $\alpha_n(t) = a_n\cos nt + b_n\sin nt$ for $n \geq 1$ and $\alpha_0(t) = a_0t + b_0$. Hence

$$u(x,t) = a_0t + b_0 + \sum_{n=1}^{\infty}[a_n\cos nt + b_n\sin nt]\cos nx$$

Now,

$$u(x,0) = b_0 + \sum_{n=1}^{\infty} a_n\cos nx = 5 + \cos 2x$$

Thus, $b_0 = 5$, $a_2 = 1$, and $a_n = 0$ for all other n.

Next,

$$u_t(x,0) = a_0 + \sum_{n=1}^{\infty} b_n\cos nx = 1$$

This tells us that $a_0 = 1$ and $b_n = 0$ for all n. The solution to our problems is therefore

$$u(x,t) = t + 5 + \cos 2t \cos 2x$$

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