Math 4581 **Quiz Two** **April 18, 2001**

You may use any books, notes, tables, calculators, or computers you wish. Be sure and write so that someone other than yourself can understand your exposition. Please do not return this sheet. *Fortuna vobiscum.*

1. Find all values of μ for which the boundary value problem

$$\varphi''(x) = \mu\varphi(x)$$
$$\varphi'(0) = \varphi'(\pi) = 0$$

has nonzero solutions, and for those values, find the corresponding solutions.

The form of the solutions to our equation $\varphi'' - \mu\varphi = 0$ depend on the sign of μ. First, suppose $\mu > 0$. Then

$$\varphi(x) = A\cosh x\sqrt{\mu} + B\sinh x\sqrt{\mu}.$$

Thus

$$\varphi'(x) = \sqrt{\mu}\left(A\cosh x\sqrt{\mu} + B\sinh x\sqrt{\mu}\right), \text{ and so}$$
$$\varphi'(0) = \sqrt{\mu}A\cosh 0 = \sqrt{\mu}A = 0.$$

We have assumed $\mu > 0$, and so we must have $A = 0$. This results in

$$\varphi'(x) = \sqrt{\mu}B\sinh x\sqrt{\mu},$$

and the other boundary condition becomes

$$\varphi'(\pi) = \sqrt{\mu}B\sinh \pi\sqrt{\mu}.$$

But this is possible only if $B = 0$. Hence there are no nonzero solutions when $\mu > 0$.

Now suppose $\mu < 0$. Then letting $\mu = -\lambda^2$ we get

$$\varphi(x) = A\cos\lambda x + B\sin\lambda x, \text{ and}$$
$$\varphi'(x) = \lambda(-A\sin\lambda x + B\cos\lambda x).$$

Next,

$$\varphi'(0) = \lambda B = 0$$

means we must have $B = 0$ since $\lambda > 0$. So,

$$\varphi'(\pi) = -\lambda\sin\lambda\pi = 0.$$

Since $\lambda \neq 0$, we must have $\lambda_n = n$, $n = \pm1, \pm2, \pm3, \ldots$. The values of μ for which there are nonzero solutions are thus

$$\mu_n = -\lambda_n^2 = n^2,\ n = 1, 2, 3, \ldots.$$

The coressponding solutions are

$$\varphi_n(x) = A\cos nx.$$

Now, what about the remaining possiblity, $\mu = 0$? Here we have

$$\varphi''(x) = 0, \text{ or}$$
$$\varphi(x) = Ax + B.$$

The requirement that $\varphi'(0) = \varphi'(\pi) = 0$ means that $A = 0$ and so $\varphi_0(x) = B$, a constant.

2. Find $u(x,t)$ such that

$$u_{xx} - u_t = 0,\ 0 < x < \pi,\ t > 0$$
$$u_x(0,t) = u_x(\pi,t) = 0$$
$$u(x,0) = f(x).$$

It should be clear from the result of Problem 1 why we assume

$$u(x,t) = \alpha_0(t) + \sum_{n=1}^{\infty} \alpha_n(t) \cos nx.$$

Thus,

$$u_{xx} - u_t = -\alpha_0'(t) + \sum_{n=1}^{\infty} [-n^2\alpha_n(t) - \alpha_n''(t)] \cos nx = 0.$$

Thus

$$-\alpha_0'(t) = 0, \text{ and}$$
$$-n^2\alpha_n(t) - \alpha_n''(t) = 0.$$

Hence,

$$\alpha_0 = a_0$$
$$\alpha_n(t) = a_n e^{-n^2 t},$$

and so

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx.$$

The constants are determined from the initial condition:

$$u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx = f(x).$$

This is simply the Fourier cosine series for f. Hence

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)dx,$$
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx,\ n = 1, 2, 3, \ldots$$

3. Find

$$\lim_{t\to\infty} u(x,t),$$

where $u(x,t)$ is the solution of the previous problem with $f(x) = x^2(x-\pi)$.

This is easy.

$$\begin{aligned}\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left[a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2 t}\cos nx \right] \\ &= a_0.\end{aligned}$$

But

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} x^2(x-\pi)dx = -\frac{\pi^3}{12}.$$

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