I. Homework handed in Monday, January 29- Solutions

1. Prove that in an inner product space

$$|f+g|^2 + |f-g|^2 = 2|f|^2 + 2|g|^2.$$

$|f+g|^2 = (f+g, f+g) = (f,f) + (f,g) + (g,f) + (g,g)$
$|f-g|^2 = (f-g, f-g) = (f,f) - (f,g)(g,f) + (g,g)$
Thus,

$$|f+g|^2 + |f-g|^2 = 2(f,f) + 2(g,g) = 2|f|^2 + 2|g|^2.$$

2. The equation in Problem 1 is called the *parallelogram equality*. Can you explain why?

3. a) Show that $B = \{e^x, e^{-x}, e^{2x}\}$ is linearly independent.

Suppose $c_1e^x + c_2e^{-x} + c_3e^{2x} = 0$. Then for $x = -1, 0, 1$, we have three equations:

$$c_1e^{-1} + c_2e^{-2} + c_3e^{-2} = 0$$
$$c_1 + c_2 + c_3 = 0$$
$$c_1e + c_2e^{-1} + c_3e^2 = 0$$

It is an easy exercise in linear algebra to show that $c_1 = c_2 = c_3 = 0$ is the only solution to the system.

b)Find the coordinates of $3\cosh x - \sinh x$ with respect to B, or show this function is not in the span of B.

$3\cosh x - \sinh x = 3\frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2} = e^x + 2e^{-x} + 0 \bullet e^{2x}$. The coordinates are thus $(1,2,0)$.

4. a) Show that $(f,g) = \int_0^\infty e^{-x}f(x)g(x)dx$ is an inner product on the space of all functions for which the integral converges.

---

i) $(f,f) = \int_0^\infty e^{-x}(f(x))^2dx \geq 0$; ii)$(f,g) = \int_0^\infty e^{-x}f(x)g(x)dx = \int_0^\infty e^{-x}g(x)f(x)dx = (g,f)$;

iii)$(\alpha f, g) = \int_0^\infty e^{-x}\alpha f(x)g(x)dx = \alpha\int_0^\infty e^{-x}f(x)g(x)dx = \alpha(f,g)$; and

iv)$(f+g,h) = \int_0^\infty e^{-x}(f(x)+g(x))h(x)dx = \int_0^\infty e^{-x}f(x)h(x)dx + \int_0^\infty e^{-x}g(x)h(x)dx$
$= (f,h) + (g,h)$.

b)Show that $\{1, 1-x, 2-4x+x^2\}$ is orthogonal with respect to the inner product defined in part **a)**.

$(1, 1-x) = \int_0^\infty e^{-x}(1-x)dx = 0$; $(1, 2-4x+x^2) = \int_0^\infty e^{-x}(2-4x+x^2)dx = 0$; and

$$(1-x, 2-4x+x^2) = \int_0^\infty e^{-x}(1-x)(2-4x+x^2)dx = 0$$

c)Find the polynomial $p(x)$ of degree ≤ 2 that minimizes

$$\int_0^\infty e^{-x}(x^3 - p(x))^2 dx.$$

We simply want the projection p of x^3 onto the space spanned by the orthogonal set $\{1, 1-x, 2-4x+x^2\}$. This projection is

$$p(x) = \frac{(x^3,1)}{(1,1)}1 + \frac{(x^3,1-x)}{(1-x,1-x)}(1-x) + \frac{(x^3,2-4x+x^2)}{(2-4x+x^2,2-4x+x^2)}(2-4x+x^2)$$

Now,

$(1,1) = \int_0^\infty e^{-x}dx = 1;$ $(1-x,1-x) = \int_0^\infty e^{-x}(1-x)^2dx = 1;$ and

$(2-4x+x^2, 2-4x+x^2) = \int_0^\infty e^{-x}(2-4x+x^2)^2dx = 4.$ Next

$(x^3,1) = \int_0^\infty e^{-x}x^3dx = 6;$ $(x^3,1-x) = \int_0^\infty e^{-x}x^3(1-x)dx = -18;$ and

$(x^3, 2-4x+x^2) = \int_0^\infty e^{-x}x^3(2-4x+x^2)dx = 36.$

Thus,

$$p(x) = \frac{6}{1}1 - \frac{18}{1}(1-x) + \frac{36}{4}(2-4x+x^2) = 6 - 18x + 9x^2$$