Homework - February 28 - Solution

Find u such that

$$\begin{aligned} u_{xx} - u_t &= 0,\ 0 \le x \le \pi,\ t > 0 \\ u(0,t) &= 0,\ u(\pi,t) = t \\ u(x,0) &= 0 \end{aligned}$$

We need a problem in which we have homogeneous boundary conditions, so let $w(x,t) = u(x,t) - \frac{x}{\pi}t$. Then we have

$$\begin{aligned} w_{xx} - w_t &= u_{xx} - u_t + \frac{x}{\pi} = \frac{x}{\pi} \\ w(0,t) &= u(0,t) = 0, \text{ and } w(\pi,t) = u(\pi,t) - t = t - t = 0. \\ w(x,0) &= u(x,0) = 0 \end{aligned}$$

As usual, we look for a solution $w(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin nx$:

$$w_{xx} - w_t = \sum_{n=1}^{\infty} \{-n^2\alpha_n(t) - \alpha_n'(t)\} \sin nx = \frac{x}{\pi}$$

Next, I hope it is clear why we need the Fouier sine series for x.

$$x = \sum_{n=1}^{\infty} b_n \sin nx,$$

$$\text{where } b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx = \frac{2}{\pi}\left(\frac{\pi(-1)^{n+1}}{n}\right) = 2\frac{(-1)^{n+1}}{n}$$

Thus,

$$\sum_{n=1}^{\infty} \{-n^2\alpha_n(t) - \alpha_n'(t)\} \sin nx = \frac{x}{\pi} = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{\pi n} \sin nx,$$

Or, making one big series:

$$\sum_{n=1}^{\infty} \left\{-n^2\alpha_n(t) - \alpha_n'(t) + 2\frac{(-1)^n}{\pi n}\right\} \sin nx = 0.$$

Now we must cope with the differential equation

$$-n^2\alpha_n(t) - \alpha_n'(t) + 2\frac{(-1)^n}{\pi n} = 0, \text{ or}$$
$$\alpha_n'(t) + n^2\alpha_n(t) = 2\frac{(-1)^n}{\pi n}.$$

To solve this, mulitply by the integrating factor $e^{n^2 t}$:

$$e^{n^2 t}[\alpha_n'(t) + n^2\alpha_n(t)] = 2\frac{(-1)^n}{\pi n}e^{n^2 t}, \text{ or}$$
$$\frac{d}{dt}\left(e^{n^2 t}\alpha_{n(t)}\right) = 2\frac{(-1)^n}{\pi n}e^{n^2 t}.$$

Thus,

$$e^{n^2 t}\alpha_{n(t)} = 2\frac{(-1)^n}{\pi n^3}e^{n^2 t} + A_n,$$

and so

$$\alpha_n(t) = 2\frac{(-1)^n}{\pi n^3} + A_n e^{-n^2 t}.$$

We're almost there:

$$w(x,t) = \sum_{n=1}^{\infty}\alpha_n(t)\sin nx$$
$$= \sum_{n=1}^{\infty}\left(2\frac{(-1)^n}{\pi n^3} + A_n e^{-n^2 t}\right)\sin nx$$

Finally, the initial condition:

$$w(x,0) = \sum_{n=1}^{\infty}\left(2\frac{(-1)^n}{\pi n^3} + A_n\right)\sin nx = 0$$

Hence,

$$A_n = -2\frac{(-1)^n}{\pi n^3},$$

and the whole gory mess is

$$w(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(1 - e^{-n^2 t}\right)\sin nx.$$

At last!

$$u(x,t) = w(x,t) + \frac{x}{\pi}t, \text{ or}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(1 - e^{-n^2 t}\right)\sin nx + \frac{x}{\pi}t, \text{ or}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\left(1 - e^{-n^2 t} - n^2 t\right)\sin nx$$

Let's take a look at some pictures. First, let's plot $u(x,t)$ for an increasing sequence of values of t: