Homework Solutions, Wednesday, March 28

Let $u(r,\theta)$ be the solution to

$$\nabla^2 u = 0, \; -\pi < \theta \le \pi, 0 \le r \le c$$
$$u(c,\theta) = g(\theta).$$

1. Show that the value of u at the center of the disc, $u(0,\theta)$, is the average of the values of u on the boundary of the disc.

We know that

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}[a_n r^n \cos n\theta + b_n r^n \sin n\theta], \text{ where}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta, \text{ and}$$

$$a_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} g(\theta)\cos n\theta d\theta, \text{ and } b_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} g(\theta)\sin n\theta d\theta$$

Thus,

$$u(0,\theta) = a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta, \text{ the average of } g \text{ on the boundary.}$$

2. Show that the value of u at the center of the disc, $u(0,\theta)$, is the average of the values of u on any circle $r = a \le c$.

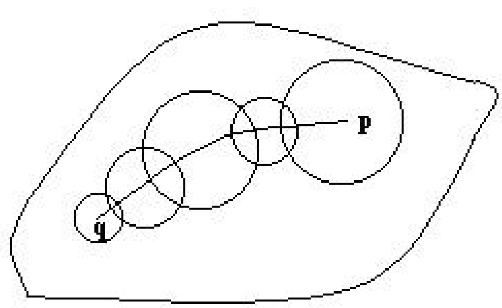
Simply compute the average of u on the circle $r = a$:

$$\begin{aligned}
\text{avg} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a,\theta)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}[a_n r^n \cos n\theta + b_n r^n \sin n\theta] \right]d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} a_0 d\theta = a_0 = u(0,\theta).
\end{aligned}$$

3. Use the result of Problem 2 to show that if $\nabla^2 u = 0$ on some region R, then the maximum and minimum values of u must occur on the boundary of u.

Suppose the maximum value of u on R occurs at some interior point **p**. Then there is a circle C centered at **p** that is completely in the interior of R. Since the maximum occurs at **p**, and we know that this maximum value is the average of the values of u on the circle, it follows that u must be constant on C (See below for a proof.). The same is true for any concentric circle of smaller radius, and so we conclude that u is constant on the disc bounded by C.

To see that u is constant on all of the region R, let **q** be another point in the interior and join **p** and **q** with an arc. We then then construct a sequence of overlapping discs, on each of which u is constant. Since the discs overlap, the constant must be the same for each disc. Thus $u(\mathbf{p}) = u(\mathbf{q})$, and we are done.



I hope it is clear how to show u is constant in case a minimum value occurs in the interior of R.

Theorem. Suppose f is a continuous function defined on the interval $[a,b]$, and let M be the maximum value of f on $[a,b]$. If the average of f on $[a,b]$ is M, then $f(x) = M$ for all $x \in [a,b]$.

Proof: If $f(x) < M$ for any point in $[a,b]$, then there is a z, with $a < z < b$ so that $f(z) < M$. But f is continuous and so there is a δ such that $f(x) < M$ for all $z - \delta < x < z + \delta$. Now,

$$\text{avg} = M = \frac{1}{b-a}\int_a^b f(x)dx$$

$$= \frac{1}{b-a}\left[\int_a^{z-\delta} f(x)dx + \int_{z-\delta}^{z+\delta} f(x)dx + \int_{z+\delta}^{b} f(x)dx\right]$$

Note that $\int_a^{z-\delta} f(x)dx \le M(z-\delta-a)$, $\int_{z-\delta}^{z+\delta} f(x)dx < M\delta$, and $\int_{z+\delta}^{b} f(x)dx \le M(b-z+\delta)$. Hence

$$
\begin{aligned}
M &= \frac{1}{b-a}\left[\int_{a}^{z-\delta} f(x)dx + \int_{z-\delta}^{z+\delta} f(x)dx + \int_{z+\delta}^{b} f(x)dx\right] \\
&< \frac{1}{b-a}[M(z-\delta-a) + M2\delta + M(b-z-\delta)] = M.
\end{aligned}
$$

Thus $M < M$, a contradiction, and so it must not be true that there is any point at which $f(x) < M$. In other words, $f(x) = M$ for all $x \in [a,b]$.