Chapter One - Linear Spaces

Definition. A **linear space** is a collection $\mathbf{V}$ of complex (real) valued functions all having the same domain such that (i) if f and g are members of $\mathbf{V}$, then so also is $f+g$; (ii) if f is a member of $\mathbf{V}$ and α is a scalar, then αf is also a member of $\mathbf{V}$. [*scalar* means complex number in case the members of $\mathbf{V}$ are complex valued and means real number in case the members of $\mathbf{V}$ are real valued.]

Example. The usual Euclidean n-space $\mathbf{R}^n$ consisting of the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers is a linear space. Here we interpret an n-tuple as a function whose domain is the set of integers $\{1, 2, \ldots, n\}$.

Exercises. Which of the following sets of functions are linear spaces?

1. All continuous real valued functions defined on the interval $[0,1]$.

2. All functions defined on the real line and having a finite number of discontinuities.

3. All functions defined on the real line and having exactly one discontinuity.

4. All functions f defined on the real line such that $f(0) = 0$.

5. All functions f defined on the real line such that $f(0) = 1$.

Definition. If $\mathbf{V}$ is a linear space and $\mathbf{W} \subset \mathbf{V}$ is also a linear space, then we say $\mathbf{W}$ is a **subspace** of $\mathbf{V}$.

Example. Let $\mathbf{V} = \mathbf{R}^2$ and let $\mathbf{W} = \{(x,y) \in \mathbf{R}^2 : 3x - 2y = 0\}$. Then $\mathbf{W}$ is a subspace of $\mathbf{V}$.

Exercises.

6. Let $\mathbf{V}$ be the set of all polynomials(real) of degree ≤ 5, and let $\mathbf{W}$ be the set of all polynomials(real) of degree ≤ 3. Is $\mathbf{W}$ a subspace of $\mathbf{V}$? Explain.

7. Describe all subspaces of $\mathbf{R}^3$.

8. Suppose $\mathbf{U}$ and $\mathbf{W}$ are both subspaces of $\mathbf{V}$. Is the intersection $\mathbf{U} \cap \mathbf{W}$ necessarily a subspace of $\mathbf{V}$? Explain.

Definition. If $f_1, f_2, \ldots, f_n$ are members of a linear space and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, then the function $\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n$ is called a **linear combination** of the functions.

Definition. Suppose $\mathbf{V}$ is a linear space and $A \subset \mathbf{V}$. The **span** of A is the set of all linear combinations of elements of A.

Proposition 1.1. If $A \subset \mathbf{V}$, then the span of A is a subspace of $\mathbf{V}$.

Definition. If $\mathbf{W}$ is a subspace of the linear space $\mathbf{V}$ and A is a subset of $\mathbf{V}$ such that $\mathbf{W}$ is the span of A, then A is said to **span $\mathbf{W}$**.

Example. In the linear space of all continuous real valued functions, the span of the functions $1, x, x^2$ is the set of all polynomials of degree ≤ 2.

Exercises.

9. In real Euclidean space $\mathbf{R}^3$, describe the span of $(1,1,1), (1,2,3)$, and $(2,3,4)$.

10. Is the function $\sin^2 x$ a member of the span of the collection $1, \cos x, \cos 2x, \cos 3x$? Explain.

11. Show that the collection of all solutions of the differential equation $y'' + 3y' + 2y = 0$ is a linear space and find a finite collection of functions which spans this space.

Definition. A set $\{f_1, f_2, \ldots, f_n\}$ of functions in a linear space is **linearly dependent** if there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that $\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n = 0$. A set that is not linearly dependent is said to be **linearly independent**. An infinite collection is said to be linearly independent if every finite subcollection is linearly independent.

Proposition 1.2. A set $\{f_1, f_2, \ldots, f_n\}$ is linearly dependent if and only if one of the functions is a linear combination of the others.

Definition. Suppose **V** is a linear space. A linearly independent set $A \subset \mathbf{V}$ that spans **V** is called a **base** for **V.**

Proposition 1.3. If the linear space **V** has a base consisting of exactly n elements, then every base for **V** consists of exactly n elements.

Definition. If the linear space **V** has a base consisting of n elements, then **V** is said to be **finite dimensional** and n is the **dimension** of **V**. A space that is not finite dimensional is said to be **infinite dimensional.**

Example. In real Euclidean space $\mathbf{R}^n$, the collection $\{(1,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,1)\}$ is a base.

Another example. The space of all real polynomials of degree $\leq n$ is finite dimensional.

Exercises.

12. Let **V** be the linear space spanned by the functions $\{e^x, e^{-x}, \cosh x, \sinh x\}$. Find a base for **V**. What is the dimension of **V**? Explain.

13. What is the dimension of the space described in Exercise 9? Explain.

14. Is the space of all real polynomials finite dimensional? Why?

Proposition 1.4. Suppose the dimension of **V** is n. Then any linearly independent collection of n elements of **V** is a base, and any collection of more than n elements is linearly dependent.

Proposition 1.5. Suppose the dimension of **V** is n. Then any collection of n elements that spans **V** is a base, any collection of fewer than n elements does not span **V**.

Proposition 1.6. Suppose $\{u_1, u_2, \ldots, u_n\}$ is a base for **V**. Then any element $v \in \mathbf{V}$ can be written uniquely as $v = \alpha_1 u_1 + \alpha_2 u_2 + \ldots, \alpha_n u_n$. The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the **coordinates** of v with respect to the given base.

Example. Let **V** be the space spanned by the set $\{e^x, e^{-x}\}$. This collection is linearly independent and so is a base for **V**. The coordinates of $\cosh x$ with respect to this base are $(1/2, 1/2)$:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Exercises.

15. a) Show that the set $B = \{1 - x^2, 1 + x^2\}$ is linearly independent.

b)Show that $5 - 2x^2$ is in the space spanned by B, and find its coordinates with respect to B.

16. Find a base for the space of all solutions of the differential equation $y'' + y = 0$.