Chapter Nine - Fourier Integrals and Transforms

Recall that for a nice periodic function f, the Fourier series

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right],$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\left(\frac{n\pi}{L}\xi\right)d\xi, \text{ and}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\left(\frac{n\pi}{L}\xi\right)d\xi$$

converges to $\frac{f(x+)+f(x-)}{2}$. ”Nice” means that f has a left hand and a right hand derivative at x. The periodic function f is said to have a discrete ”spectrum.” The so-called spectrum of f consists of those values of $\frac{n\pi}{L}$ for which either of the coefficients a_n or b_n is not zero. Thus the spectrum of f is discrete. A nice nonperiodic function can be represented in essentially the same way by an integral superposition of cosines and sines:

$$f \sim \int_0^{\infty}[a_\lambda \cos\lambda x + b_\lambda \sin\lambda x]d\lambda,$$

where

$$a_\lambda = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\lambda\xi d\xi, \text{ and}$$

$$b_\lambda = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin\lambda\xi d\xi.$$

This is the **Fourier integral** for f. In this case, ”nice” must include also the requirement that the integral $\int_{-\infty}^{\infty}|f(\xi|d\xi$ be finite. In this case, f has a ”continuous spectrum.” Just as with the Fourier series, the integral converges to $\frac{f(x+)+f(x-)}{2}$ at points where f has both a left and right hand derivative. Analogous to the cosine and sine series, we have the Fourier cosine integral

$$f \sim \int_0^\infty a_\lambda \cos \lambda x dx, \text{ with}$$

$$a_\lambda = \frac{2}{\pi} \int_0^\infty f(\xi) \cos \lambda \xi d\xi;$$

and the sine integral

$$f \sim \int_0^\infty a_\lambda \sin \lambda x dx, \text{ with}$$

$$a_\lambda = \frac{2}{\pi} \int_0^\infty f(\xi) \sin \lambda \xi d\xi.$$

The cosine integral is, of course, simply the Fourier integral of the function $\hat{f}$ that is the even extension of f to the entire real line; and the sine integral, the Fourier integral of the odd extension of f.

Exercises

1. Find the Fourier integral of the function f given by

$$f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

2. Sketch the graph of the limit of the integral found in Problem 1.

To see the strong analogy between the Fourier series and the Fourier integral, consider the following differential equation:

$$\begin{aligned} u_{xx} - u_t &= 0, x > 0, t > 0 \\ u(0,t) &= 0, u \text{ bounded as } x \to \infty; \\ u(x,0) &= f(x). \end{aligned}$$

As usual, consider the attendant eigenvalue problem

$$\begin{aligned} \varphi''(x) &= \mu\varphi(x) \\ \varphi(0) &= 0, \varphi \text{ bounded as } x \to \infty. \end{aligned}$$

For $\mu > 0$, we have

$$\varphi(x) = A\cos\lambda x + B\sinh\lambda x, \text{ where } \mu = \lambda^2.$$

Now, the condition $\varphi(0) = 0$ means we must have $A = 0$. But $\sinh\lambda x$ is not bounded as $x \to \infty$, and so there are no nonzero solutions for $\mu > 0$.

Next, suppose $\mu < 0$. The solutions to the equation are now

$$\varphi(x) = A\cos\lambda x + B\sin\lambda x, \text{ where } \mu = -\lambda^2.$$

Then the requirement that $\varphi(0) = 0$ tells us that $\varphi(x) = B\sin\lambda x$. This solution is perfectly well-behaved as $x \to \infty$, so we have that λ can be any positive number.

Finally, what if $\mu = 0$? The $\varphi(x) = Ax + B$ and again there are no nonzero solutions. In summary, any $\lambda > 0$ is an eigenvalue with corresponding eigenfunction $\varphi_\lambda(x) = \sin\lambda x$. We thus assume a solution of original problem

$$u(x,t) = \int_0^\infty \alpha_\lambda(t)\sin\lambda x d\lambda.$$

This gives us

$$u_{xx} - u_t = \int_0^\infty [-\lambda^2\alpha_\lambda(t) - \alpha'_\lambda(t)]\sin\lambda x dx = 0.$$

In other words, $\alpha_\lambda(t) = a_\lambda e^{-\lambda^2 t}$, and we have

$$u(x,t) = \int_0^\infty a_\lambda e^{-\lambda^2 t}\sin\lambda x d\lambda.$$

The function a_λ is determined from the initial condition

$$u(x,0) = \int_0^\infty a_\lambda \sin\lambda x d\lambda = f(x)$$

to be

$$a_\lambda = \frac{2}{\pi}\int_0^\infty f(\xi)\sin\lambda\xi d\xi.$$

Exercises

3. Solve

$$\begin{aligned} u_{xx} - u_t &= 0,\ x > 0,\ t > 0 \\ u(0,t) &= 0,\ u \text{ bounded as } x \to \infty. \\ u(x,0) &= \begin{cases} 1 & 0 \le x \le \pi \\ 0 & x > \pi. \end{cases} \end{aligned}$$

4. Solve

$$\begin{aligned} u_{xx} - u_{tt} &= 0,\ x > 0,\ t > 0 \\ u(0,t) &= 0,\ u \text{ bounded as } x \to \infty. \\ u(x,0) &= f(x), \text{ and } u_t(x,0) = g(x). \end{aligned}$$

We look next at the so-called complex forms of the Fourier series and integral. Specifically, we write the cosine and sine as complex exponentials. First, in the Fourier series, let

$$\begin{aligned} \cos\left(\frac{n\pi}{L}x\right) &= \frac{e^{i\frac{n\pi}{L}x} + e^{-i\frac{n\pi}{L}x}}{2}, \text{ and} \\ \sin\left(\frac{n\pi}{L}x\right) &= \frac{e^{i\frac{n\pi}{L}x} - e^{-i\frac{n\pi}{L}x}}{2i}. \end{aligned}$$

This results in

$$\begin{aligned} f &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n - ib_n}{2} e^{i\frac{n\pi}{L}x} + \frac{a_n + ib_n}{2} e^{-i\frac{n\pi}{L}x} \right]. \end{aligned}$$

If we set $c_0 = \frac{a_0}{2}$, and

$$c_n = \left(\frac{a_n - ib_n}{2}\right), \text{ and } c_{-n} = \left(\frac{a_n + ib_n}{2}\right),$$

then we have

$$f \sim \sum_{-\infty}^{\infty} c_n e^{i\frac{n\pi}{L}x}.$$

Observe that for $n = 1, 2, 3, \ldots$, we have

$$c_n = \left(\frac{a_n - ib_n}{2}\right) = \frac{1}{2L}\int_{-L}^{L} f(\xi)\left(\cos\frac{n\pi}{L}\xi - i\sin\frac{n\pi}{L}\xi\right)d\xi$$

$$= \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{-i\frac{n\pi}{L}\xi}d\xi.$$

Also,

$$c_{-n} = \left(\frac{a_n + ib_n}{2}\right) = \frac{1}{2L}\int_{-L}^{L} f(\xi)\left(\cos\frac{n\pi}{L}\xi + i\sin\frac{n\pi}{L}\xi\right)d\xi$$

$$= \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{i\frac{n\pi}{L}\xi}d\xi.$$

Thus for all $n = 0, \pm 1 \pm 2, \ldots,$ we may write

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(\xi)e^{-i\frac{n\pi}{L}\xi}d\xi.$$

In essentially the same manner, we get the complex form of the Fourier integral:

$$f \sim \int_{-\infty}^{\infty} c_\lambda e^{i\lambda x}dx, \text{ where}$$

$$c_\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)e^{-i\lambda\xi}d\xi.$$

The function c_λ is traditionally called the **Fourier transform** of *f*.

[To be continued]