Chapter Two - Inner Product Spaces

Definition. Suppose **V** is a linear space. An **inner product** on **V** is a scalar valued function F from pairs of elements of **V** such that
i) $F(f,f) \geq 0$.
ii) $F(f,g) = \overline{F(g,f)}$,
iii) $F(\alpha f,g) = \alpha F(f,g)$,
iv) $F(f+g,h) = F(f,h) + F(g,h)$.

Note. Where there is no danger of confusion, we shall write simply (f,g) for $F(f,g)$.

Proposition 2.1. An inner product has the following properties:
i) $(f,\alpha g) = \overline{\alpha}(f,g)$ *ii*) $(f,g+h) = (f,g) + (f,h)$

Example. In real Euclidean n-space, define $(u,v) = \sum_{j=1}^{n} u_j v_j$ where $u = (u_1,u_2,\ldots,u_n)$ and $v = (v_1,v_2,\ldots,v_n)$. Then (u,v) is an inner product.

Exercises.

1. Prove that in complex n-space, $(u,v) = \sum_{j=1}^{n} u_j \overline{v}_j$ is an inner product.
$[u = (u_1,u_2,\ldots,u_n)$ and $v = (v_1,v_2,\ldots,v_n)]$.
2. In the space of all continuous functions of period 2π, prove that

$$(f,g) = \int_0^{2\pi} f(t)\overline{g(t)}dt$$

is an inner product.
3. Prove Proposition 2.1.

Definition. Suppose **V** is an inner product space and $f \in$ **V**. The **norm** of f is the real number $|f| = \sqrt{(f,f)}$.

Proposition 2.2. $|(f,g)| \leq |f||g|$.

Note. The inequality in Proposition 2.2 is the celebrated **Cauchy-Schwarz-Buniakovsky Inequality.**

Example. Let $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ be members of ordinary everyday Euclidean 3-space. Then, of course, $(u,v) = u_1v_1 + u_2v_2 + u_3v_3$ is the usual inner product, and we remember from Mrs. Turner's 3^{rd} grade calculus that $(u,v) = |u||v|\cos\varphi$, where φ is the angle between u and v. In this case, the $C-S-B$ inequality says simply that $|\cos\varphi| \leq 1$.

Proposition 2.3. The norm satisfies the following properties:
i) $|f| \geq 0,$
ii) $|\alpha f| = |\alpha||f|,$
iii) $|f+g| \leq |f| + |g|.$

Note. The inequality *iii*) in Proposition 2.3 is called the **Minkowski Inequality** or sometimes the **triangle inequality.**

Example. Again, in ordinary Euclidean 3-space with the usual inner product, $|u|$ is the actual honest-to-goodness measure-it-with-a-meter-stick length of the vector u, and Proposition 2.3 gives obvious geometric facts. You can see why the Minkowski Inequality is frequently called the triangle inequality.

Definition. Elements f and g in an inner product space are said to be **orthogonal** if $(f,g) = 0$.

Example. In real Euclidean 3-space with the usual inner product, u and v are orthogonal if and only if the vectors are perpendicular.

Exercise.

4. Prove that if $(f,g) = 0$, then $|f+g|^2 = |f|^2 + |g|^2$. (This is called the **Pythagorean Theorem**. Why?)

Definition. A sequence of elements $\{\varphi_1, \varphi_2, \dots\}$ is called an **orthogonal sequence** if $|\varphi_k| \neq 0$ for all k, and $(\varphi_i, \varphi_j) = 0$ for all $i \neq j$.

Theorem 2.4. Every orthogonal sequence is linearly independent.

Theorem 2.5. If $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is an orthogonal base for a linear space **V** and $f \in$ **V**, then

$$f = \sum_{i=1}^{n} \alpha_i \varphi_i$$

where

$$\alpha_i = \frac{(f, \varphi_i)}{(\varphi_i, \varphi_i)}.$$

Corollary 2.6. $|f|^2 = \sum_{i=1}^{n} |a_i|^2.$

Exercise.

5. Let **V** be the space of all real continuous functions defined on the interval $[-1,1]$, and let the inner product (f,g) be defined by

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx.$$

a) Show that the collection $B = \{1, x, 3x^2 - 1, 5x^3 - 3x\}$ is orthogonal.
b) Let **W** be the subspace of **V** spanned by B. Find the coordinates of $2x^3 + x^2$ with respect to the base B.

Theorem 2.7. Suppose **H** is a finite dimensional subspace of an inner product space **V**. Suppose in addition that $f \in \mathbf{V}$. Let $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ be an orthogonal base for **H**. Then there is exactly one element $g \in \mathbf{H}$ such that

$$|f - g|^2 \leq |f - h|^2 \text{ for all } h \in \mathbf{H}.$$

Moreover,

$$g = \sum_{i=1}^{n} \alpha_i \varphi_i, \text{ where } \alpha_i = \frac{(f, \varphi_i)}{(\varphi_i, \varphi_i)}.$$

Corollary 2.8. $\left| f - \sum_{i=1}^{n} \alpha_i \varphi_i \right|^2 = |f|^2 - \sum_{i=1}^{n} |\alpha_i|^2 |\varphi_i|^2$

Definition. The function g in Theorem 2.6 is called the **projection** of f onto **H**, and is usually denoted $proj(f : \varphi_1, \varphi_2, \ldots, \varphi_n)$

Theorem 2.9. If $h \in \mathbf{H}$, then $(f - proj(f : \varphi_1, \varphi_2, \ldots, \varphi_n), h) = 0$.

Example. Let **V** be the space of all continuous real functions defined on the interval $[-1, 1]$ with the inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx.$$

Let **H** be the subspace of **V** consisting of all polynomials of degree ≤ 2. Then it is easy to verify that $\{1, x, 3x^2 - 1\}$ is an orthogonal base for **H**. The projection of $\cos\frac{\pi x}{2}$ onto **H** is then given by

$$g = \frac{(\cos\frac{\pi x}{2}, 1)}{(1,1)} 1 + \frac{(\cos\frac{\pi x}{2}, x)}{(x,x)} x + \frac{(\cos\frac{\pi x}{2}, 3x^2 - 1)}{(3x^2 - 1, 3x^2 - 1)} (3x^2 - 1).$$

Let's do the calculations:

$(\cos\frac{\pi x}{2}, 1) = \int_{-1}^{1} \cos\frac{\pi x}{2} dx = \frac{4}{\pi}$; $\qquad (\cos\frac{\pi x}{2}, x) = \int_{-1}^{1} x \cos\frac{\pi x}{2} dx = 0$;

$(\cos\frac{\pi x}{2}, 3x^2 - 1) = \int_{-1}^{1} (3x^2 - 1)\cos\frac{\pi x}{2} dx = 8\frac{\pi^2 - 12}{\pi^3}$;

$(1,1) = \int_{-1}^{1} dx = 2$; and $(3x^2 - 1, 3x^2 - 1) = \int_{-1}^{1} (3x^2 - 1)^2 dx = \frac{8}{5}$.

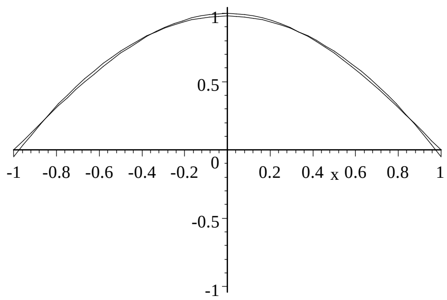
Thus,

$$g(x) = \frac{2}{\pi} + 5\frac{\pi^2 - 12}{\pi^3}(3x^2 - 1).$$

According to Theorem 2.7, this is the quadratic that best fits $\cos\frac{\pi x}{2}$ in the sense that it is the quadratic that among all quadratics minimizes

$$\int_{-1}^{1}(\cos\frac{\pi x}{2} - p(x))^2 dx.$$

Let's plot both g and $\cos\frac{\pi x}{2}$ on the same set of axes:



Note the graphs are almost indistinguishable!

Theorem 2.10. Let $\{u_1, u_2, \dots, u_n\}$ be a base for the space **H**. Then an orthogonal base for **H** is $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$, where

$$\begin{aligned}
\varphi_1 &= u_1, \\
\varphi_2 &= u_2 - proj(u_2 : \varphi_1) \\
\varphi_3 &= u_3 - proj(u_3 : \varphi_1, \varphi_2) \\
&\vdots \\
\varphi_i &= u_i - proj(u_i : \varphi_1, \varphi_2, \dots, \varphi_{i-1}) \\
&\vdots \\
\varphi_n &= u_n - proj(u_n : \varphi_1, \varphi_2, \dots, \varphi_{n-1})
\end{aligned}$$

Exercises.

6. Find the polynomial of degree ≤ 3 that best fits $\cos\frac{\pi x}{2}$ in the sense that among all such polynomials $p(x)$, it minimizes

$$\int_{-1}^{1}(\cos\frac{\pi x}{2} - p(x))^2 dx.$$

[See Exercise 5.]

7. In Euclidean 3-space, find the point in the plane $2x+y-3z = 0$ that is closest to the point $(0,0,5)$.

8. Let **V** be the collection of all functions with period 2π with the inner product

$$(f,g) = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

a) Verify that the collection $C = \{\sin x, \sin 2x, \sin 3x\}$ is orthogonal.

b) Let *f* be the periodic extension of

$$\hat{f}(x) = \begin{cases} -1, \text{ for } -\pi < x \le 0 \\ 1, \text{ for } 0 < x \le \pi \end{cases},$$

and find the projection of *f* onto the space spanned by the collection *C*.

c) Draw the graph of *f* and the projection found in **b)** on the same axes.

Definition. An orthonormal sequence $\{\varphi_1, \varphi_2, \ldots\}$ in which $|\varphi_i|^2 = 1$ for each i is called an **orthonormal** sequence.

Proposition 2.11. Suppose $\{\varphi_1, \varphi_2, \ldots\}$ is an orthonormal sequence. Then for $\alpha_k = (f, \varphi_k)$, the sequence $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges, and

$$\sum_{k=1}^{\infty} |\alpha_k|^2 \le |f|^2.$$

[This is the celebrated **Bessel's inequality**.]

Corollary 2.12. $\lim_{k\to\infty} \alpha_k = 0.$

[This one is known as **Riemann's Lemma**.]

Definition. A linearly independent sequence $B = \{u_1, u_2, \ldots\}$ of elements in an inner product space **V** is called an **approximating base** for **V** if for every element $f \in \mathbf{V}$, given $\varepsilon > 0$ there is an element u of the span of B so that $|f-u| < \varepsilon$.

Definition. A sequence (f_n) in an inner product space is said to **converge in the mean** to f if $\lim_{n\to\infty} |f_n - f|^2 = 0$.

Proposition 2.13. Suppose $\{\varphi_1, \varphi_2, \ldots\}$ is an orthonormal approximating base for **V**, and $f \in \mathbf{V}$. Then the series

$$\sum_{k=1}^{\infty} \alpha_k \varphi_k$$

where $\alpha_k = (f, \varphi_k)$ converges in the mean to f.