## Chapter Three - Fourier Series

**Definition.** If $f$ is integrable on the interval $[-\pi,\pi]$, the series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt dt, \text{ and}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt dt$$

is called the **Fourier Series** associated with $f$.

**Proposition 3.1.** With respect to the inner product $(f,g) = \int_{-\pi}^{\pi} f(x)g(x)dx$, the sequence

$$T = \{1,\cos x,\sin x,\cos 2x,\sin 2x,\cos 3x,\sin 3x,\ldots\}$$

is an orthogonal sequence.

**Exercise.**

**1.** In the space of nice functions with the inner product defined in Proposition 3.1, find the projection of $f$ onto the space spanned by $T_n = \{1,\cos x,\sin x,\ldots,\cos nx,\sin nx\}$.

**Definition.** Suppose $f$ is a real valued function and $a$ is a real number. If the derivative $f'$ exists in an interval $[a-\varepsilon,a)$ and the limit of $f'(x)$ as $x\to a$ from the left exists, we call this limit the **left-hand derivative** of $f$ at $a$. This limit is traditionally denoted by $f'(a-)$.

**Exercise.**

**2.** Guess the definition of the right-hand derivative $f'(a+)$ of a function $f$ at at the point $a$.

**Theorem 3.1.** If $f$ is a function of period $2\pi$, then its Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ at every point $x$ at which it has both a right-hand and a left-hand derivative.

**Example.** Let's find the Fourier series for the function $f(x)$ on the interval $[-\pi,\pi]$ :

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos kt dt = 0,$$

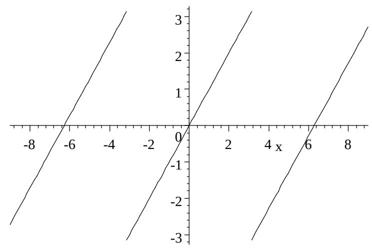
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin kt\,dt = \frac{1}{k^2}(\sin kt - kt\cos kt)\Big|_{-\pi}^{\pi}$$

$$= -\frac{2}{k}\cos k\pi = (-1)^{k+1}\frac{2}{k}$$

The Fourier series is thus

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx$$

Now, what does the limit of this series look like? We simply apply Theorem 3.1 to the periodic extension of *f*. This results in




**Exercises**

**4.** Find the Fourier series for $f(x) = x^2$ on the interval $[-\pi,\pi]$ and sketch the graph of its limit.

**5.** Let

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Find the Fourier series for *f* on the interval $[-\pi,\pi]$ and sketch the graph of its limit.

**Definitions.** If *f* is integrable on the interval $[0,\pi]$, the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos kx$$

where

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx\,dx$$

is called the **Fourier cosine series** for *f*. The series

$$\sum_{k=1}^{\infty} b_k \sin kx$$

where

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\,dx$$

is called the **Fourier sine series** for *f*.

**Observations.** The Fourier cosine series for *f* is the Fourier series for $\tilde{f}$, where

$$\tilde{f}(x) = \begin{cases} f(-x) & -\pi \le x < 0 \\ f(x) & 0 \le x \le \pi \end{cases}.$$

The function $\tilde{f}$ is called the **even extension** of *f* to the interval $[-\pi, \pi]$.
The Fourier sine series for *f* is the Fourier series for $\tilde{f}$, where

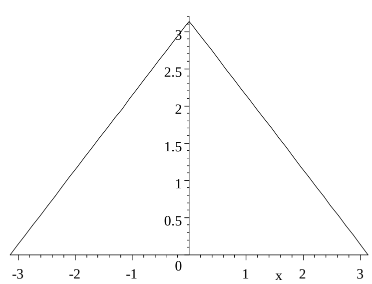
$$\tilde{f}(x) = \begin{cases} -f(-x) & -\pi \le x < 0 \\ f(x) & 0 \le x \le \pi \end{cases}.$$

The function $\tilde{f}$ is called the **odd extension** of *f* to the interval $[-\pi, \pi]$.

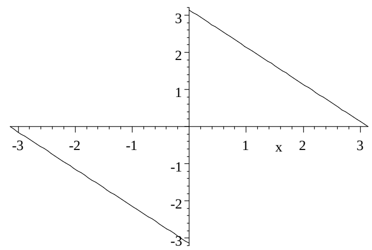
**Example.** Let $f(x) = \pi - x$, for $0 \le x \le \pi$. The even extension to the interval $[-\pi, \pi]$ is then

$$\tilde{f}(x) = \begin{cases} \pi + x & \text{if } -\pi \le x < 0 \\ \pi - x & \text{if } 0 \le x \le \pi \end{cases}$$

A picture:




The odd extension looks like




**Exercise**

**6.** Find the cosine series for $f(x) = \pi - x$, for $0 \le x \le \pi$ and sketch the graph of its limit on the interval $[-3\pi, 3\pi]$.

**7.** Find the sine series for $f(x) = \pi - x$, for $0 \le x \le \pi$ and sketch the graph of its limit on the interval $[-3\pi, 3\pi]$.