Chapter Four - Differential Equations I

Consider the problem of finding $u(x,t)$ for $0 \le x \le \pi$ and $t \ge 0$, such that $u_{xx}(x,t) - u_t(x,t) = 0$, subject to the conditions $u(0,t) = u(\pi,t) = 0$, and $u(x,0) = f(x)$. This partial differential equation is an example of the celebrated **heat equation**. A physical model for the problem here is the temperature distribution at time t in a rod of length π in which the ends are constantly held at 0°, and the initial temperature distribution $u(x,0)$ is specified to be $f(x)$.

Recall that for a linear function (or "operator") in finite dimensional spaces, there are times when we can find a base for our space with respect to which the operator becomes very simple. Specifically, if the linear operator is represented by the matrix **A**, if there is an orthogonal base consisting of elements $\{v_1, v_2, \ldots, v_n\}$ such that $\mathbf{A}v_j = \mu_j v_j$, then for any element $x = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$, we have $\mathbf{A}x = \mu_1 a_1 v_1 + \mu_2 a_2 v_2 + \ldots + \mu_n a_n v_n$ The numbers μ_j are usually called **eigenvalues** and the corresponding vectors v_j are called **eigenvectors**, or **eigenfunctions**. Now, what does this have to do with our problem at hand? Well, here we seek something similar in the infinite dimensional case for the linear operator $\mathbf{L}\varphi = \varphi''$ on the space of all twice differentiable functions such that $\varphi(0) = \varphi(\pi) = 0$. Our original partial differential equation can then be turned into a simple easy to solve collection of ordinary differential equations. Let's see what we're talking about.

Consider the problem of finding nonzero φ such that $\mathbf{L}\varphi = \mu\varphi$, where **L** is the linear operator on the space of twice differentiable functions on $[0,\pi]$ which vanish at 0 and π defined by $\mathbf{L}\varphi = \varphi''$. We are thus faced with the boundary value problem

$$\begin{aligned} \varphi'' - \mu\varphi &= 0, \\ \varphi(0) = \varphi(\pi) &= 0 \end{aligned}$$

We recall from Mrs. Turner's calculus class that any solution of this equation looks like

$$\varphi(x) = Ae^{x\sqrt{\mu}} + Be^{-x\sqrt{\mu}}.$$

If $\sqrt{\mu}$ is real (in other words, if $\mu \ge 0$, then $\varphi(0) = \varphi(\pi) = 0$ gives us the two equations

$$\begin{aligned} A + B &= 0, \text{ and} \\ Ae^{\pi\sqrt{\mu}} + Be^{-\pi\sqrt{\mu}} &= 0. \end{aligned}$$

Or,

$$A(e^{\pi\sqrt{\mu}} - e^{-\pi\sqrt{\mu}}) = 2A\sinh\pi\sqrt{\mu} = 0.$$

This can happen only if $A = 0$, or $\sinh\pi\sqrt{\mu} = 0$. If $A = 0$, then $B = 0$ also, and we have no nonzero solution. If $\sinh\pi\sqrt{\mu} = 0$, then it must be true that $\pi\sqrt{\mu} = 0$, or, $\mu = 0$. This also results in only the zero solution. We see then that for $\mu > 0$, there are no nonzero solutions to our problem. So what if $\mu < 0$? Let's see. First, for convenience and to remind us that $\mu < 0$, let $\mu = -\lambda^2$. Then any solution of the differential equation is

$$\varphi(x) = A\cos\lambda x + B\sin\lambda x.$$

The boundary conditions become

$$\varphi(0) = A = 0, \text{ and}$$
$$\varphi(\pi) = B\sin\lambda\pi = 0.$$

Again, $B = 0$ results in only the zero solution, and so we must have $\sin\lambda\pi = 0$. Thus, $\lambda\pi = n\pi$, where $n = \pm1, \pm2, \ldots$. (Note that $n = 0$ does not give a nonzero solution to our differential equation.) We have now found eigenvalues $\mu_n = -n^2$, and corresponding eigenfunctions $\varphi_n(x) = \sin nx$. We have found an infinite collection of eigenfunctions, and from our vast knowledge of Fourier series, we know the collection is orthogonal with respect to the inner product $(f,g) = \int_0^\pi f(x)g(x)dx$.

Now back to our original partial differential equation. We think first of the variable t are a parameter and write $u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin nx$. The equation becomes

$$u_{xx} - u_t = \sum_{n=1}^{\infty} -n^2\alpha_{n(t)}\sin nx - \sum_{n=1}^{\infty}\alpha_n'(t)\sin nx = 0, \text{ or}$$
$$= \sum_{n=1}^{\infty}[-n^2\alpha_{n(t)} - \alpha_n'(t)]\sin nx = 0.$$

Now we need to have $-n^2\alpha_{n(t)} - \alpha_n'(t) = 0$, or

$$\alpha_n'(t) = -n^2\alpha_{n(t)}$$

The solution to this is easy:

$$\alpha_n(t) = a_n e^{-n^2 t}, \text{ where } a_n \text{ is any constant.}$$

Putting this back in the expression for u gives us

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t}\sin nx.$$

What are the constants a_n? Simple! They come from the initial condition $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = f(x),$$

and we see that the a_n are simply the Fourier sine coefficients for f:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx.$$

Example. For the problem

$$\begin{aligned} u_{xx} - u_t &= 0,\ 0 \le x \le \pi,\ t \ge 0; \\ u(0,t) &= u(\pi,t) = 0, \\ u(x,0) &= f(x),\ \text{where} \end{aligned}$$

$$f(x) = \begin{cases} x & x < \pi/2 \\ \pi - x & \pi/2 < x \end{cases}$$

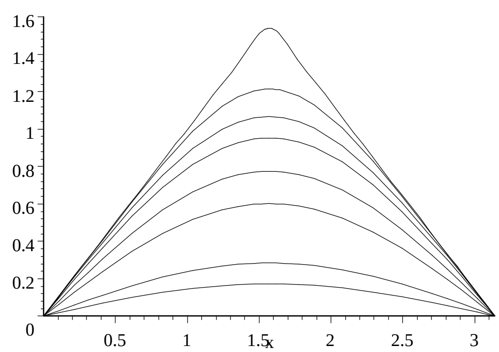
Then we have

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx = \frac{4}{\pi}\frac{\sin\frac{1}{2}n\pi}{n^2}$$

Letting $n = 2k - 1$ gives us

$$u(x,y) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}e^{-(2k-1)^2 t}\sin(2k-1)x$$

Let's draw $u(x,t)$ for a sequence of values of t starting at $t = 0$:



Looks like what one would expect!

Exercises

1. Solve

$$\begin{aligned} u_{xx} - u_t &= 0,\ \ 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= x(\pi - x), \end{aligned}$$

and sketch the graph of $u(x,t)$ for a few values of t.

What do we do if the boundary conditions are not homogeneous? In other words, suppose that instead of specifying that $u(0,t) = u(\pi,t) = 0$, we want to have

$$u(0,t) = A(t), \text{ and } u(\pi,t) = B(t).$$

The answer is remarkably simple. We define $v(x,t) = A(t)(1 - x/\pi) + B(t)x$, and let

$$w(x,t) = u(x,t) - v(x,t).$$

Then

$$w_{xx} - w_t = u_{xx} - v_{xx} - (u_t - v_t) = v_t.$$

Also,

$$\begin{aligned} w(0,t) &= u(0,t) - v(0,t) = A(t) - A(t) = 0, \text{ and} \\ w(\pi,t) &= u(\pi,t) - v(\pi,t) = B(t) - B(t) = 0. \end{aligned}$$

Also,

$$w(x,0) = u(x,0) - v(x,0) = f(x) - v(x,0) = g(x)$$

We have thus turned our problem with nonhomogeneous boundary conditions into one with homogeneous boundary conditions but with a nonzero ”source term” :

$$w_{xx} - w_t = v_t$$

We look at an example of how this works. Just as before, we let $w(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin nx$:

$$u_{xx} - u_t = \sum_{n=1}^{\infty} -n^2 \alpha_{n(t)} \sin nx - \sum_{n=1}^{\infty} \alpha_n'(t) \sin nx = v_t = A'(t)(1 - x/\pi) + B'(t)x.$$

$$\sum_{n=1}^{\infty}[-n^2\alpha_{n(t)} - \alpha_n'(t)]\sin nx = \sum_{n=1}^{\infty} b_n(t)\sin x, \text{ or}$$

$$\sum_{n=1}^{\infty}[-n^2\alpha_{n(t)} - \alpha_n'(t) - b_n(t)]\sin nx = 0$$

Again, we have a differential equation

$$\alpha_n'(t) + n^2\alpha_n(t) = b_n(t)$$

What are the functions $b_n(t)$? Well, we want

$$v_t(x,t) = A'(t)(1 - x/\pi) + B'(t)x = \sum_{n=1}^{\infty} b_n(t)\sin nx.$$

Or,

$$A'(t) + (B'(t) - A'(t)/\pi)x = \sum_{n=1}^{\infty} b_n(t)\sin nx$$

The Fourier sine series for the function 1 is

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin nx,$$

and for x is

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-\pi}{n}\cos n\pi \sin nx$$

Thus,

$$A'(t) + (B'(t) - A'(t)/\pi)x = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[A'(t)\frac{1-\cos n\pi}{n} + (B'(t) - A'(t))\frac{-\pi}{n}\cos n\pi\right]\sin nx.$$

Hence,

$$b_n(t) = A'(t)\frac{1-\cos n\pi}{n} + (B'(t) - A'(t))\frac{-\pi}{n}\cos n\pi$$

Example. Consider the problem

$$\begin{aligned} u_{xx} - u_t &= 0,\ 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) &= 0,\ \ u(\pi,t) = \sin t \\ u(x,0) &= 0 \end{aligned}$$

Let $v(x,t) = \frac{x}{\pi}\sin t,$ and so we have $w(x,t) = u(x,t) - \frac{x}{\pi}\sin t$. Then

$$\begin{aligned} w_{xx} - w_t &= u_{xx} - u_t + \frac{x}{\pi}\cos t = \frac{x}{\pi}\cos t \\ w(0,t) &= u(0,t) = 0, \text{ and } w(\pi,t) = u(\pi,t) - \sin t = \sin t - \sin t = 0. \\ w(x,0) &= u(x,0) = 0 \end{aligned}$$

As usual, $w(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin nx$:

$$w_{xx} - w_t = \sum_{n=1}^{\infty} \{-n^2\alpha_n(t) - \alpha'_n(t)\}\sin nx = \frac{x}{\pi}\cos t$$

Next, I hope it is clear why we need the Fourier sine series for x.

$$x = \sum_{n=1}^{\infty} b_n \sin nx,$$

$$\text{where } b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx dx = \frac{2}{\pi}\left(\frac{\pi(-1)^{n+1}}{n}\right) = 2\frac{(-1)^{n+1}}{n}$$

Thus,

$$\sum_{n=1}^{\infty} \{-n^2\alpha_n(t) - \alpha'_n(t)\}\sin nx = \frac{x}{\pi}\cos t = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{\pi n}\cos t \sin nx,$$

Or, making one big series:

$$\sum_{n=1}^{\infty}\left\{-n^2\alpha_n(t) - \alpha'_n(t) + 2\frac{(-1)^n}{\pi n}\cos t\right\}\sin nx = 0.$$

Now we must cope with the differential equation

$$\begin{aligned} -n^2\alpha_n(t) - \alpha'_n(t) + 2\frac{(-1)^n}{\pi n}\cos t &= 0, \text{ or} \\ \alpha'_n(t) + n^2\alpha_n(t) &= 2\frac{(-1)^n}{\pi n}\cos t. \end{aligned}$$

To solve this, multiply by the integrating factor $e^{n^2 t}$:

$$e^{n^2t}[\alpha_n'(t) + n^2\alpha_n(t)] = 2\frac{(-1)^n}{\pi n}e^{n^2t}\cos t, \text{ or}$$
$$\frac{d}{dt}\left(e^{n^2t}\alpha_{n(t)}\right) = 2\frac{(-1)^n}{\pi n}e^{n^2t}\cos t.$$

Thus,

$$e^{n^2t}\alpha_{n(t)} = 2\frac{(-1)^n}{\pi n}\left(\frac{n^2}{n^4+1}e^{n^2t}\cos t + \frac{1}{n^4+1}e^{n^2t}\sin t\right) + A_n,$$

and so

$$\alpha_n(t) = 2\frac{(-1)^n}{\pi n(n^4+1)}(n^2\cos t + \sin t) + A_n e^{-n^2t}.$$

We're almost there:

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty}\alpha_n(t)\sin nx \\ &= \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{\pi n(n^4+1)}(n^2\cos t + \sin t) + A_n e^{-n^2t}\right)\sin nx \end{aligned}$$

Finally, the initial condition:

$$w(x,0) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{\pi n(n^4+1)}(n^2) + A_n\right)\sin nx = 0$$

Hence,

$$A_n = -\frac{2n^2(-1)^n}{\pi n(n^4+1)},$$

and the whole gory mess is

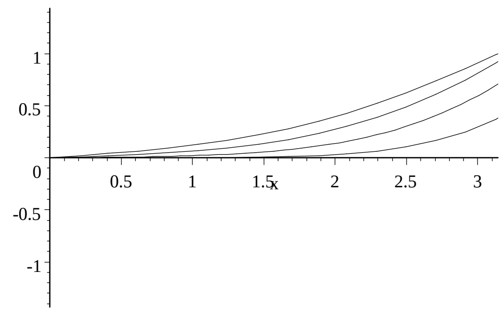
$$w(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^4+1)}\left(n^2(\cos t - e^{-n^2t}) + \sin t\right)\sin nx.$$

At last!

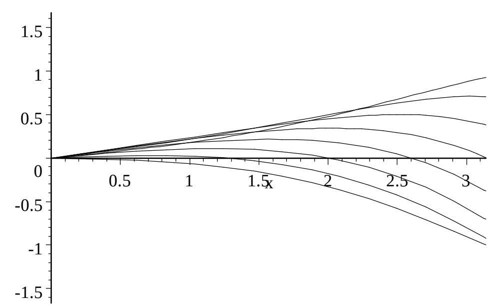
$$u(x,t) = w(x,t) + \frac{x}{\pi}\sin t, \text{ or}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^4+1)}\left(n^2(\cos t - e^{-n^2 t}) + \sin t\right)\sin nx + \frac{x}{\pi}\sin t$$

Let's take a look at some pictures. First, let's plot $u(x,t)$ for a sequence of values of t between 0 and $\pi/2$:



Take a look now at some pictures for t between $\pi/2$ and $3\pi/2$:
$u(x, 12\pi/8)$



No surprises, I hope.

Exercises

2. Solve

$$u_{xx} - u_t = 0, 0 \le x \le \pi, \ t \ge 0$$
$$u(0,t) = 1, \ u(\pi,t) = 10$$
$$u(x,0) = 0.$$

Sketch the solution for a few values of t.

3. Solve

$$\begin{aligned} u_{xx} - u_t &= 0, 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) &= 0,\ u(\pi,t) = t \\ u(x,0) &= 0. \end{aligned}$$

Sketch the solution for a few values of t.

Observation. It should be clear now how to handle a problem in which there is a source term: $u_{xx} - u_t = F(x,t)$, *etc.*

Exercise

4. Solve

$$\begin{aligned} u_{xx} - u_t &= t\sin x, 0 \le x \le \pi,\ t \ge 0 \\ u(0,t) &= 0,\ u(\pi,t) = 0 \\ u(x,0) &= 0. \end{aligned}$$