Chapter Five - Eigenvalues, Eigenfunctions, and All That

The partial differential equation methods described in the previous chapter is a special case of a more general setting in which we have an equation of the form

$$\mathbf{L}_1(x)u(x,t) + \mathbf{L}_2(t)u(x,t) = F(x,t)$$

for $x \in D$ and $t \geq 0$, in which u is specified on the boundary of D as are initial conditions at $t = 0$. We assume that $\mathbf{L}_1(x)$ is a linear differential operator in x and $\mathbf{L}_2(t)$ is a linear differential operator in t.

In the previous chapter, $\mathbf{L}_1(x)\varphi(x) = \varphi''(x)$, and we were fortunate that this operator had an infinite orthogonal collection of eigenfunctions which approximate functions in a nice way. We shall see what happens in a more general setting. First a bit of linear algebra review.

Definition. Suppose V is a linear space together with an inner product. A linear operator $L : V \to V$ such that $(Lf,g) = (f,Lg)$ for all $f,g \in V$ is said to be **self-adjoint.**

Let V be a linear space of nice functions defined on an interval $a \leq x \leq b$ with the inner product $(f,g) = \int_a^b f(x)\overline{g(x)}dx$. Define the linear operator L by

$$L\varphi = \frac{d}{dx}\left[p(x)\frac{d\varphi}{dx}\right] + q(x)\varphi(x).$$

We assume that p and q are real and continuous and that p is continuously differentiable and positive. Such an operator is called a **Sturm-Liouville** operator.

Definition. Suppose r is a real continuous and positive function on $a \leq x \leq b$. A scalar μ such that $L\varphi = -\mu r\varphi$ for some nonzero $\varphi \in V$ is called an **eigenvalue** of L, and the function φ is an **eigenfunction.**

For reasons that will soon be clear, we would very much like to have our linear operator L be self-adjoint. Thus, we want $(Lf,g) = (f,Lg)$ for all f and g in the space V. Let's compute:

$$\begin{aligned}(Lf,g) - (f,Lg) &= \int_a^b \left[Lf(x)\overline{g(x)} - f(x)\overline{Lg(x)}\right]dx \\ &= \int_a^b \left\{\frac{d}{dx}\left[p(x)\frac{df}{dx}\right]\overline{g(x)} + q(x)f(x)\overline{g(x)} - \right. \\ &\qquad \left. \frac{d}{dx}\left[p(x)\frac{d\overline{g}}{dx}\right]f(x) - q(x)\overline{g(x)}f(x)\right\}dx\end{aligned}$$

$$= \int_a^b \left\{ \frac{d}{dx}\left[p(x)\frac{df}{dx} \right]\overline{g(x)} - \frac{d}{dx}\left[p(x)\frac{d\overline{g}}{dx} \right] f(x) \right\} dx$$

$$= \int_a^b \frac{d}{dx}\left\{ p(x)\left[\overline{g(x)}\frac{df}{dx} - f(x)\frac{d\overline{g}}{dx} \right] \right\} dx$$

$$= p(b)\overline{g(b)}f'(b) - f(b)\overline{g'(b)} - p(a)\left[\overline{g(a)}f'(a) - f(a)\overline{g'(a)}(a) \right].$$

In other words, L will be self-adjoint if we choose V to be a linear space of functions such that

$$p(b)\left[\overline{g(b)}f'(b) - f(b)\overline{g'(b)} \right] = p(a)\left[\overline{g(a)}f'(a) - f(a)\overline{g'(a)}(a) \right].$$

Examples. The operator $L\varphi = \varphi''$ is a Sturm-Liouville operator on the interval $a \leq x \leq b$, with $p(x) = 1$ and $q(x) = 0$. The self-adjoint boundary conditions then become

$$\overline{g(b)}f'(b) - f(b)\overline{g'(b)} = \overline{g(a)}f'(a) - f(a)\overline{g'(a)}(a)$$

Thus, if we choose V to be the space of functions such that $\varphi(a) = \varphi(b) = 0$, the operator L is self-adjoint.

So why should we care about self-adjoint operators? Let's see.

Proposition 1. Every eigenvalue of a self-adjoint operator is real.

Proof: Suppose $L\varphi = -\mu r\varphi$ for some nonzero φ. Then $(L\varphi, \varphi) = (\varphi, L\varphi)$ because L is self-adjoint, and this becomes $(-\mu r\varphi, \varphi) = (\varphi, -\mu r\varphi)$, or $\mu(r\varphi, \varphi) = \overline{\mu}(\varphi, r\varphi)$. In other words,

$$\mu \int_a^b r(x)|\varphi(x)|^2 dx = \overline{\mu} \int_a^b r(x)|\varphi(x)|^2 dx.$$

The integral is not zero, and so we have $\mu = \overline{\mu}$.

Proposition 2. Eigenfunctions corresponding to different eigenvalues of a self-adjoint operator L are orthogonal with respect to the inner product

$$(f,g)_r = (rf, g) = (f, rg) = \int_a^b r(x)f(x)\overline{g(x)}dx.$$

Proof: Suppose $L\varphi = -\mu r\varphi$ and $L\xi = -\nu r\xi$ for $\mu \neq \nu$. Again, from the fact that L is self-adjoint, we know that $(L\varphi, \xi) = (\varphi, L\xi)$. Thus, $(-\mu r\varphi, \xi) = (\varphi, -\nu r\xi)$. From the previous proposition we know that μ and ξ are real. Thus,

$$\mu(r\varphi,\xi) = \nu(\varphi,r\xi), \text{ or}$$

$$(\mu-\nu)(\varphi,\xi)_r = 0.$$

But $\mu \neq \nu$ and so it must be true that $(\varphi,\xi)_r = 0$.

Nothing we have done guarantees the existence of eigenvalues of L. There are, in fact, an infinite number of them, giving an infinite orthogonal set of eigenfunctions. A proof is too much for these notes. The celebrated **Sturm-Liouville** Theorem says even more. It says that every nice function *f* can be expanded in a series

$$f = \sum_{n=1}^{\infty} a_n \varphi_n$$

of the eigenfunctions φ_n, and this series converges in the mean to *f*.

Example. Consider the problem

$$u_{xx} - u_t = 0, \text{ for } 0 < x < \pi \text{ and } t > 0$$

$$u(0,t) = 0 \text{ and } u_t(\pi,t) = 0$$

$$u(x,0) = f(x)$$

In solving this problem, I hope it is clear why we begin with the eigenvalue problem

$$\varphi'' = -\mu\varphi$$

$$\varphi(0) = 0,\ \varphi'(\pi) = 0.$$

Note this is an example of the Sturm-Liouville problem we have just discussed, so we know what to expect. First, we know that we must $\mu \geq 0$ in order to have nonzero solutions to the problem. To remind us of that, let $\mu = \lambda^2$. Then

$$\varphi(x) = A\cos\lambda x + B\sin\lambda x,$$

and the boundary conditions become

$$\varphi(0) = A = 0$$

and

$$\varphi'(0) = \lambda B \cos \lambda\pi = 0.$$

If λ or B are zero, then $\varphi = 0$, so it must be true that $\cos \lambda\pi = 0$. Thus $\lambda\pi$ must be an odd multiple of $\pi/2$:

$$\lambda_n = (2n-1)\frac{1}{2} = (n - \frac{1}{2}), \text{ for } n = 1, 2, 3, \ldots$$

Our eigenvalue are thus $\mu_n = \lambda_n^2 = (n - \frac{1}{2})^2$, and he corresponding eigenfunctions are

$$\varphi_n(x) = \sin\left(n - \frac{1}{2}\right)x.$$

Continuing, let $u(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin \lambda_n x$. From here on, things look familiar.

$$u_{xx} - u_t = \sum_{n=1}^{\infty} [-\mu_n \alpha_n(t) - \alpha_n'(t)] \sin \lambda_n x = 0.$$

As usual,

$$-\mu_n \alpha_n(t) - \alpha_n'(t) = 0$$

tells us that $\alpha_n(t) = a_n e^{-\mu_n t} = a_n e^{-\lambda_n^2 t}$, and so

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \sin \lambda_n x.$$

The constants a_n are determined from the initial condition $u(x,0) = f(x)$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin \lambda_n x.$$

Thus,

$$\begin{aligned} a_n &= \frac{(f, \sin \lambda_n x)}{(\sin \lambda_n x, \sin \lambda_n x)} \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin \lambda_n dx \end{aligned}$$

Exercises

1. Find all eigenvalues and eigenfunctions for

$$\varphi'' = -\mu\varphi,$$
$$\varphi(0) = \varphi'(\pi) = 0.$$

2. Find all eigenvalues and eigenfunctions:

$$\varphi'' = -\mu\varphi,$$
$$\varphi(0) = \varphi(\pi)$$
$$\varphi'(0) = \varphi'(\pi)$$

Let n be a non-negative integer, and consider the Sturm-Liouville problem

$$\frac{d}{dx}\left[x\frac{d\varphi}{dx}\right] - \frac{n^2}{x}\varphi(x) = -\lambda^2 x\varphi(x),\ 0 < x < b$$
$$\varphi(c) = 0.$$

Note that this operator does not quite fit our definitions. The coefficient $q(x) = -\frac{n^2}{x}$ of φ is not nice at the left end point of the interval under consideration and $p(x) = x$ is zero at the left end point of the interval. This is what is known as a *singular* Sturm-Liouville problem. If, however, we consider functions that, together with their derivatives, are reasonably well-behave at the left end point $a = 0$, then our self-adjoint boundary conditions

$$p(b)\left[\overline{g(b)}f'(b) - f(b)\overline{g'(b)}\right] = p(a)\left[\overline{g(a)}f'(a) - f(a)\overline{g'(a)}(a)\right]$$

become

$$c\left[\overline{g(b)}f'(b) - f(b)\overline{g'(b)}\right] = 0,$$

and are satisfied by functions in the space of those nice functions that vanish at $x = b$.

The equation

$$\frac{d}{dx}\left[x\frac{d\varphi}{dx}\right] - \frac{n^2}{x}\varphi(x) = -\lambda^2 x\varphi(x), \text{ or}$$
$$x^2\varphi'' + x\varphi' - n^2\varphi + \lambda^2 x^2\varphi = 0$$

is the world-famous **Bessel's Equation.** We shall seek solutions by means of the Frobenius method. That is, assume a solution φ of the form

$$\varphi = x^{\alpha} \sum_{k=0}^{\infty} c_k x^k, \; c_0 \neq 0.$$

Then

$$\begin{aligned}
\varphi &= \sum_{k=0}^{\infty} c_k x^{\alpha+k}, \\
\varphi' &= \sum_{k=0}^{\infty} (\alpha+k) c_k x^{\alpha+k-1}, \\
\varphi'' &= \sum_{k=0}^{\infty} (\alpha+k)(\alpha+k-1) c_k x^{\alpha+k-2},
\end{aligned}$$

and we have

$$\begin{aligned}
x^2\varphi'' + x\varphi' - n^2\varphi + \lambda^2 x^2 \varphi = & \sum_{k=0}^{\infty} [(\alpha+k)(\alpha+k-1) + (\alpha+k) - n^2] c_k x^{\alpha+k} + \\
& \sum_{k=0}^{\infty} \lambda^2 c_k x^{\alpha+k+2}
\end{aligned}$$

Writing $\sum_{k=0}^{\infty} \lambda^2 c_k x^{\alpha+k+2} = \sum_{k=2}^{\infty} \lambda^2 c_{k-2} x^{\alpha+k}$ gives us

$$\begin{aligned}
x^2\varphi'' + x\varphi' - n^2\varphi + \lambda^2 x^2 \varphi &= \sum_{k=0}^{\infty} [(\alpha+k)^2 - n^2] c_k x^{\alpha+k} + \sum_{k=2}^{\infty} \lambda^2 c_{k-2} x^{\alpha+k} \\
&= (\alpha^2 - n^2) c_0 x^{\alpha} + [(\alpha+1)^2 - n^2] c_1 x^{\alpha+1} + \\
&\quad \sum_{k=0}^{\infty} \{[(\alpha+k)^2 - n^2] c_k + \lambda^2 c_{k-2}\} x^{\alpha+k}.
\end{aligned}$$

In order for the differential equation to be satisfied, each of the coefficients of the powers of x must be zero:

$$\begin{aligned}
(\alpha^2 - n^2) c_0 &= 0 \\
[(\alpha+1)^2 - n^2] c_1 &= 0 \\
[(\alpha+k)^2 - n^2] c_k + \lambda^2 c_{k-2} &= 0, \text{ for } k = 2,3,4,\ldots.
\end{aligned}$$

In the first equation, $c_0 \neq 0$ and so it must be true that $\alpha^2 - n^2 = 0$, giving us $\alpha = n$, or $-n$. Let’s start with $\alpha = n$. The remainder of the equations become

$$(2n+1)c_1 = 0$$
$$k(2n+k)c_k + \lambda^2 c_{k-2} = 0, \text{ for } k = 2,3,4,\ldots.$$

Thus, $c_1 = c_3 = c_5 = \ldots = 0$, and

$$c_k = \frac{-\lambda^2}{k(2n+k)} c_{k-2}, \text{ for } k = 2,3,4,\ldots.$$

Letting $k = 2m$ results in

$$c_{2m} = \frac{-\lambda^2}{2m(2n+2m)} c_{2(m-1)} = \frac{-\lambda^2}{2^2 m(n+m)} c_{2(m-1)}.$$

Then

$$c_{2m} = \frac{-\lambda^2}{2^2 m(n+m)} c_{2(m-1)} = \frac{(-1)^m \lambda^{2m} n!}{2^{2m} m!(n+m)!} c_0$$

where c_0 is any constant. For the sake of neatness, choose $c_0 = \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n$. Then

$$c_{2m} = \frac{(-1)^m \lambda^{2m} n!}{2^{2m} m!(n+m)!} c_0 = \left(\frac{\lambda}{2}\right)^n \frac{(-1)^m}{m!(n+m)!} \left(\frac{\lambda}{2}\right)^{2m},$$

and our solutions, at last,

$$\varphi(x) = \left(\frac{\lambda}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{\lambda}{2}\right)^{2m} x^{2m+n}, \text{or}$$
$$\varphi(x) = \left(\frac{\lambda x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{\lambda x}{2}\right)^{2m}.$$

The function

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m}$$

is the **Bessel function of the first kind of order n.** Our solution is thus

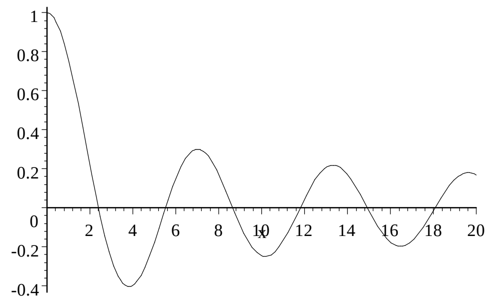
$$\varphi(x) = J_n(\lambda x).$$

A second linearly independent solution is found reducing the order of the original equation. It turns out not to be nice at $x = 0$, and so we have solutions $\varphi(x) = AJ_n(\lambda x)$. The boundary condition is thus

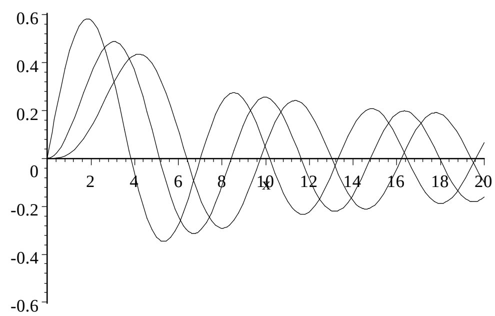
$$J_n(\lambda b) = 0.$$

It is known that J_n has an infinite number of non-negative zeros. These and all sorts of other information about Bessel functions can be found in most handbooks or by means of most computer algebra systems; *e.g.*, *Maple* or *Mathematica*. Here are some pictures.

First, J_0 :



Now here are $J_1, J_2,$ and J_3 :



It is clear then that our eigenvalues are λ_{nm}^2, with $\lambda_{nm} = z_{nm}/b$, where z_{nm} is the m^{th} zero of J_n. Here is a short table of some of these values:

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$
1	2.405	3.832	5.135	6.379
2	5.520	7.016	8.417	9.760
3	8.654	10.173	11.620	13.017
4	11.792	13.323	14.796	16.224

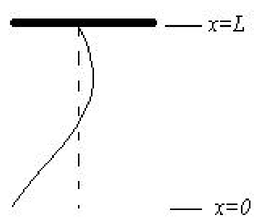
The corresponding eigenfunctions are, of course, $J_n(\lambda_{nm}x)$. Note there is a non-constant weight function here, so our orthogonality becomes

$$\int_0^b xJ_n(\lambda_{nk}x)J_n(\lambda_{nl}x)dx = 0, \text{ for } k \neq l.$$

Example. Consider the problem

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - u_{tt} = 0,\ 0 < x < L,\ t > 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x),\ u_{tt}(x,0) = g(x).$$

The solution u gives the displacement of a hanging chain of length L :



It should be clear by now why we are interested in the eigenvalue problem

$$\frac{d}{dx}\left(x\frac{d\varphi}{dx}\right) = -\lambda^2\varphi$$
$$\varphi(L) = 0, \text{ and } \varphi(0) \text{ nice.}$$

This is tantalizingly close to Bessel's equation of order zero. We could use the method of Frobenius to find solutions, *etc.* Instead, let's make a change of variable. Let $z = 2\sqrt{x}$, or $x = z^2/4$. Define $\phi(z) = \varphi(z^2/4)$. Then,

$$\frac{d\phi}{dz} = \frac{z}{2}\frac{d\varphi}{dx}, \text{ or } \frac{d\varphi}{dx} = \frac{2}{z}\frac{d\phi}{dz}.$$

Our differential equation thus becomes

$$\frac{d}{dx}\left(x\frac{d\varphi}{dx}\right) + \lambda^2\varphi = \frac{2}{z}\frac{d}{dz}\left(\frac{z^2}{4}\frac{2}{z}\frac{d\phi}{dz}\right) + \lambda^2\phi = 0, \text{ or}$$
$$\frac{d}{dz}\left(z\frac{d\phi}{dz}\right) + \lambda^2 z = 0.$$

Now we have Bessel's equation with $n = 0$. The solution is $J_0(\lambda z) = J_0(2\lambda\sqrt{x})$. The boundary condition $\varphi(L) = 0$ is $J_0(2\lambda\sqrt{L}) = 0$, and so we have the eigenvalues

$$\lambda_m = \frac{z_m}{2\sqrt{L}},$$

where z_m is the m^{th} zero of J_0.

We set

$$u(x,t) = \sum_{m=1}^{\infty} \alpha_m(t) J_0(2\lambda_m\sqrt{x}).$$

Then

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - u_{tt} = \sum_{m=1}^{\infty}(-\lambda_m^2\alpha_m(t) - \alpha_m''(t))J_0(2\lambda_m\sqrt{x}) = 0.$$

This gives us $\alpha_m(t) = a_m\cos\lambda_m t + b_m\sin\lambda_m t$. Thus,

$$u(x,t) = \sum_{m=1}^{\infty}(a_m\cos\lambda_m t + b_m\sin\lambda_m t)J_0(2\lambda_m\sqrt{x})$$

Use the initial conditions to find the constants a_m, b_m :

$$u(x,0) = \sum_{m=1}^{\infty} a_m J_0(2\lambda_m\sqrt{x}) = f(x).$$

Hence,

$$a_m = \frac{\int_0^L f(x)J_0(2\lambda_m\sqrt{x})dx}{\int_0^L [J_0(2\lambda_m\sqrt{x})]^2 dx}.$$

Also,

$$u_t(x,0) = \sum_{m=1}^{\infty} b_m\lambda_m J_0(2\lambda_m\sqrt{x}),$$

and so,

$$b_m = \frac{\int_0^L g(x) J_0(2\lambda_m \sqrt{x})\,dx}{\lambda_m \int_0^L [J_0(2\lambda_m \sqrt{x})]^2\,dx}$$

Exercise

3. Find the first four terms of the series for u in case $L = 1$, $f(x) = x(1 - x)$, and $g(x) = 0$. Draw some pictures of your approximation for different values of t.