Chapter Six - Laplace’s Equation

Laplace’s equation, or the potential equation, is $\nabla^2 u = 0$, where the operator ∇^2 is the **Laplacian**, the divergence of the gradient. In rectangular coordinates in two dimensions,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

We are interested in the problem of finding u on a region R such that $\nabla^2 u = 0$ in the interior of R and u is some specified function on the boundary of R. This is called the **Dirichlet** problem. We begin with a very special example:

$$\begin{aligned}\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0,\ 0 < x,y < \pi \\ u(0,y) = u(\pi,y) &= 0, \text{ and} \\ u(x,0) = f(x),\ u(x,\pi) &= g(x).\end{aligned}$$

From all that has gone before, it should be clear why we set

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n(y) \sin nx.$$

This gives us

$$\sum_{n=1}^{\infty} [-n^2 \alpha_n(y) + \alpha_n''(y)] \sin nx = 0,$$

which leads to the ordinary differential equation

$$-n^2 \alpha_n(y) + \alpha_n''(y) = 0.$$

From this, we conclude that

$$\alpha_n(y) = a_n \cosh ny + b_n \sinh ny.$$

Thus,

$$u(x,y) = \sum_{n=1}^{\infty} [a_n \cosh ny + b_n \sinh ny] \sin nx.$$

The remaining boundary conditions lead to

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx = f(x), \text{ and}$$

$$u(x,\pi) = \sum_{n=1}^{\infty} [a_n \cosh n\pi + b_n \sinh n\pi] \sin nx = g(x).$$

Hence we need

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx, \text{ and}$$

$$a_n \cosh n\pi + b_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} g(x) dx, \text{ or}$$

$$b_n = \frac{1}{\sinh n\pi} \left[\frac{2}{\pi} \int_0^{\pi} g(x) dx - a_n \cosh n\pi \right]$$

Example. Consider the problem with $f(x) = x(\pi - x)$ and $g(x) = 0$. Then

$$a_n = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx dx$$
$$= \frac{4}{\pi n^3} [1 + (-1)^{n+1}]$$

$$b_n = -a_n \frac{\cosh n\pi}{\sinh n\pi}.$$

Now we have

$$u(x,y) = \sum_{n=1}^{\infty} \frac{a_n}{\sinh n\pi} [\cosh ny \sinh n\pi - \sinh ny \cosh n\pi] \sin nx$$
$$= \sum_{n=1}^{\infty} \frac{a_n}{\sinh n\pi} \sinh(n(\pi - y)) \sin nx.$$

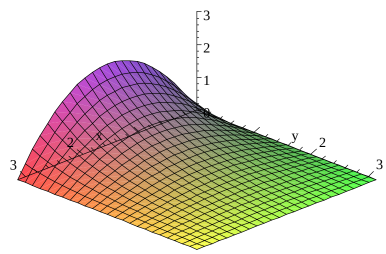
We can simplify things a bit by observing that $a_n = 0$ for n even. Let $n = 2k - 1$. Then

$$a_{2k-1} = \frac{8}{\pi(2k-1)^3},$$

and we have

$$u(x,y) = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{\sinh((2k-1)(\pi-y))}{(2k-1)^3\sinh(2k-1)\pi}\sin(2k-1)x.$$

Here is a picture:



Exercise

1. Show that the solution to the original problem can be written

$$u(x,y) = \sum_{n=1}^{\infty}\left[\frac{A_n\sinh(n(\pi-y)) + B_n\sinh ny}{\sinh n\pi}\right]\sin nx, \text{ where}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx, \text{ and } B_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin nx dx$$

2. Find the solution for $f(x) = 0$ and $g(x) = x(\pi - x)$.

Now, what do we do about more realistic boundary conditions; *viz*, those in which u does not have to be zero on $x = 0$ and $x = \pi$? The answer is rather simple. Suppose we want to have $u(x,0) = f(x)$, $u(x,\pi) = g(x)$, $u(0,y) = h(y)$, and $u(\pi,y) = k(y)$. We first find the solution of the problem in case $h(y) = k(y) = 0$. This we have already done. The solution $v(x,y)$ is given above. Next, we find the solution $w(x,y)$ of the problem with $f(x) = g(x) = 0$, and $h(y)$ and $k(y)$ given. Note there is really nothing new here. This is just the previous problem with x and y interchanged–we simply turn our heads. The solution u of the general problem is then $u(x,y) = v(x,y) + w(x,y)$.

Example. Let's solve the general problem with $f(x) = x^2$, $g(x) = 0$, $k(y) = (\pi - y)^2$, and $h(y) = 0$. First, v.

$$v(x,y) = \sum_{n=1}^{\infty}\left[\frac{A_n\sinh(n(\pi-y))}{\sinh n\pi}\right]\sin nx, \text{ where}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx dx = \frac{2}{\pi n^3}[(-1)^{n+1}n^2\pi^2 + 2((-1)^n - 1)].$$

Or,

$$v(x,y) = \frac{2}{\pi}\sum_{n=1}^{40} \frac{(-1)^{n+1}n^2\pi^2 + 2((-1)^n - 1)}{n^3 \sinh n\pi} \sinh(n(\pi - y)) \sin nx$$

Next,

$$w(x,y) = \sum_{n=1}^{\infty}\left[\frac{B_n \sinh nx}{\sinh n\pi}\right]\sin ny, \text{ where}$$

$$B_n = \frac{2}{\pi}\int_0^{\pi}(\pi - y)^2 \sin ny dy = \frac{2}{\pi}\left[\frac{2((-1)^n - 1) + n^2\pi^2}{n^3}\right]$$

:

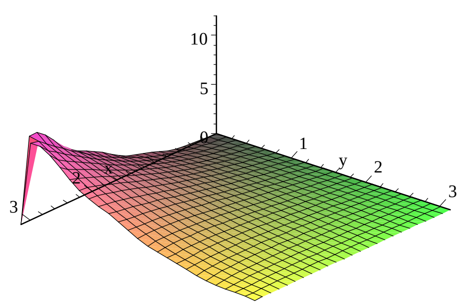
Or,

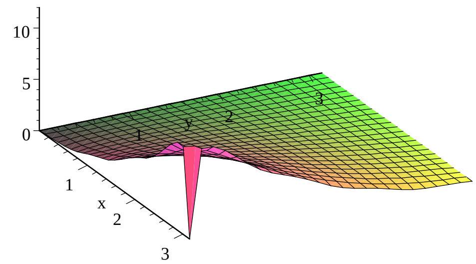
$$w(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}(2((-1)^n - 1) + n^2\pi^2)\frac{\sinh nx \sin ny}{n^3 \sinh n\pi}$$

At last,

$$u(x,y) = v(x,y) + w(x,y)$$

Here are a couple of views of the graph of *u*.





Note that it looks fairly nice, except at the corner $(0,\pi)$ of the region. Our problem came with nice continuous boundary conditions, but we then split the problem into two problems each of which has discontinuous boundary conditions. Thus both v and w, and hence u, are zero at this corner. We are seeing the nasty behavior of the trigonometric series at discontinuities (Gibb's phenomenon).Could this have been avoided? Yes indeed–we simply replace u by $U = u - V$, where the function v is chosen to insure that U is zero at all four corners of the rectangular region.

How do we find such a V? Easy. We let $V(x,y) = a_1 + a_2x + a_3y + a_4xy$, where the a_i are determined so that $V = U$ at the corners. Let's see how this works with the problem we just completed. In this case, we want

$$\begin{aligned} V(0,0) &= a_1 = 0, \\ V(0,\pi) &= a_3\pi = 0, \\ V(\pi,\pi) &= a_2\pi + a_4\pi^2 = 0, \text{ and} \\ V(\pi,0) &= a_2\pi = \pi^2. \end{aligned}$$

I hope it is clear that $V(x,y) = \pi x - xy = x(\pi - y)$ does the job. We thus consider the problem

$$\begin{aligned} \nabla^2 U &= \nabla^2(u - V) = \nabla^2 u = 0, \\ U(x,0) &= u(x,0) - V(x,0) = x^2 - x\pi \\ U(\pi,y) &= (\pi - y)^2 - \pi(\pi - y) = -y(\pi - y), \\ U(0,y) &= 0 - 0 = 0, \text{ and } U(x,\pi) = 0 - 0. \end{aligned}$$

Study the solution just given and observe U has the same form as the solution to that problem, except we need

$$A_n = \frac{2}{\pi}\int_0^{\pi} x(x-\pi)\sin nx\,dx = \frac{4}{\pi n^3}((-1)^n - 1), \text{ and}$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} y(y-\pi)\sin ny\,dy = \frac{4}{\pi n^3}((-1)^n - 1)$$

Thus,

$$v(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}((-1)^n - 1)\frac{\sinh(n(\pi - y))}{n^3 \sinh n\pi}\sin nx, \text{ and}$$

$$w(x,y) = \frac{4}{\pi}\sum_{n=1}^{\infty}((-1)^n - 1)\frac{\sinh nx}{n^3 \sinh n\pi}\sin ny$$

Then

$$\begin{aligned} U(x,y) &= v(x,y) + w(x,y) \\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n^3 \sinh n\pi}(\sinh(n(\pi - y))\sin nx + \sinh nx \sin ny). \end{aligned}$$

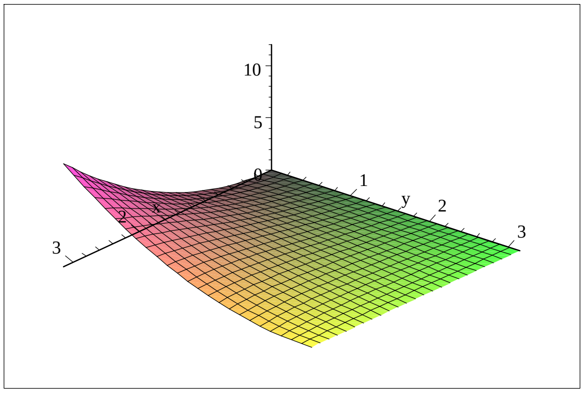
Finally,

$$u(x,y) = U(x,y) + V(x,y) = U(x,y) + x(\pi - y).$$

Or,

$$u(x,y) = x(\pi - y) + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n^3 \sinh n\pi}(\sinh(n(\pi - y))\sin nx + \sinh nx \sin ny).$$

This is, of course, precisely the same as the solution found before, but this one should have better convergence properties. Let’s take a look at a picture of the first 30 terms of this series:



This one is absolutely gorgeous at the corner $(\pi, 0)$! Oooh...aahh.]

Exercise

3. Solve

$$\nabla^2 u = 0,\ 0 < x,y) < \pi$$
$$u(0,y) = \left(y - \frac{\pi}{2}\right)^2,\ u(x,\pi) = \left(x - \frac{\pi}{2}\right)^2$$
$$u(x,0) = u(\pi,y) = 0.$$

Consider Laplace's equation

$$\nabla^2 u = 0$$

on the disc of radius *a* and centered at the origin. Specifically, consider the problem

$$\nabla^2 u = 0 \text{ for } x^2 + y^2 \le c^2,$$
$$u = f \text{ on the boundary } x^2 + y^2 = c^2.$$

In polar coordinates, the Laplacian operator looks like

$$\nabla^2 u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Thus we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$
$$u(c,\theta) = g(\theta).$$

I hope it is clear from all that has gone before that we should consider the eigenvalue problem

$$\frac{d^2\varphi}{d\theta^2} = -\lambda^2\varphi$$
$$\varphi(\pi) = \varphi(-\pi), \text{ and}$$
$$\varphi'(\pi) = \varphi'(-\pi)$$

From our vast knowledge of Sturm-Liouville problems, we know what to expect. Let's see what we get.

$$\varphi(\theta) = A\cos\lambda\theta + B\sin\lambda\theta$$

and so our boundary conditions become

$$A\cos\lambda\pi + B\sin\lambda\pi = A\cos(-\lambda\pi) + B\sin(-\lambda\pi), \text{ and}$$
$$\lambda[-A\sin\lambda\pi + B\cos\lambda\pi] = \lambda[A\sin(-\lambda\pi) - B\cos(-\lambda\pi)].$$

Or,

$$2B\sin\lambda\pi = 0$$
$$\lambda A\sin\lambda\pi = 0$$

A moment's reflection should convince you that we obtain eigenvalues $\lambda_n^2 = n^2$ for $n = 0, 1, 2\ldots$. Corresponding to the eigenvalue $\lambda_0^2 = 0$, we have the eigenfunction $\varphi(\theta) = 1$, and corresponding to each eigenvalue $\lambda_n^2 = n^2$, we have two independent eigenfunctions $\cos n\theta$ and $\sin n\theta$.

With

$$u(r,\theta) = \alpha_0(r) + \sum_{n=1}^{\infty}[\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta]$$

we have

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \\
&= \frac{1}{r}\frac{d}{dr}(\alpha_0'(r)) + \sum_{n=1}^{\infty}[\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r))\cos n\theta + \frac{1}{r}\frac{d}{dr}(r\beta_n'(r))\sin n\theta \\
&\quad - n^2\frac{\alpha_n(r)}{r^2}\cos n\theta - n^2\frac{\beta_n(r)}{r^2}\sin n\theta] \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dr}(\alpha_0'(r)) \\
&+ \sum_{n=1}^{\infty}[\left(\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r)) - n^2\frac{\alpha_n(r)}{r^2}\right)\cos n\theta + \left(\frac{1}{r}\frac{d}{dr}(r\beta_n'(r)) - n^2\frac{\beta_n(r)}{r^2}\right)\sin n\theta \\
&= 0.
\end{aligned}$$

This gives us the differential equations

$$\frac{1}{r}\frac{d}{dr}(r\alpha_0'(r)) = 0,$$
$$\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r)) - \frac{1}{r^2}n^2\alpha_n(r) = 0, \text{ and}$$

$$\frac{1}{r}\frac{d}{dr}(r\beta_n'(r)) - \frac{1}{r^2}n^2\beta_n(r) = 0.$$

The first one is easy: $r\alpha_0'(r) = A$. Thus, $\alpha_0(r) = A\log r + B$. The requirement that the solution be nice at $r = 0$ means that A must be 0. Thus α_0 =constant = a_0. Next,

$$\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r)) - \frac{1}{r^2}n^2\alpha_n(r) = 0 \text{ becomes}$$
$$r^2\alpha_n''(r) + r\alpha_n'(r) - n^2\alpha_n(r) = 0.$$

This, as you no doubt remember from Mrs. Turner's calculus class, is a so-called Cauchy-Euler equation, all solutions of which are

$$\alpha_n(r) = Ar^n + Br^{-n}.$$

Again, the solutions must be nice at $r = 0$, and so $B = 0$, and our solutions are

$$\alpha_n(r) = a_n r^n.$$

In exactly the same way, we get

$$\beta_n(r) = b_n r^n.$$

Putting it all together gives us

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}[a_n r^n \cos n\theta + b_n r^n \sin n\theta].$$

The condition $u(c,\theta) = g(\theta)$ becomes

$$g(\theta) = a_0 + \sum_{n=1}^{\infty}[a_n c^n \cos n\theta + b_n c^n \sin n\theta].$$

Thus,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta,$$

$$a_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} g(\theta)\cos n\theta d\theta, \text{ and}$$

$$b_n = \frac{1}{\pi c^n} \int_{-\pi}^{\pi} g(\theta) \sin n\theta d\theta$$

Example. Suppose $c = 1$ and $g(\theta) = \theta^2$. Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 d\theta = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos n\theta d\theta = \frac{4(-1)^n}{n^2}, \text{ and}$$

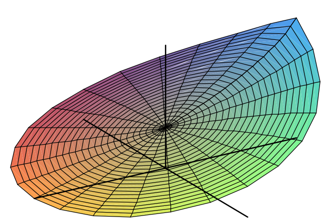
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \sin n\theta d\theta = 0$$

Hence,

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} [a_n r^n \cos n\theta + b_n r^n \sin n\theta]$$

$$u(r,\theta) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{30} \frac{(-1)^n}{n^2} r^n \cos n\theta$$

Here is a picture:



Exercises

4. a)Show that the value of u at the center of the disc, $u(0,\theta)$, is the average of the values of u on the boundary of the disc.

b)Show that the value of u at the center of the disc, $u(0,\theta)$, is the average of the values of u on any circle $r = a \leq c$.

5. a)Use the result of Problem 4 to show that if $\nabla^2 u = 0$ on some region R, then the maximum value of u occurs on the boundary of u only if u is constant on R.

b)Show that if $\nabla^2 u = 0$ on some region R, then the minimum value of u occurs on the boundary of u only if u is constant on R.