## Chapter Seven - Vibrating Strings

The displacement $u$ of a string is described by the equation

$$\frac{\partial}{\partial x}\left(T(x)\frac{\partial u}{\partial x}\right) - \rho(x)\frac{\partial^2 u}{\partial u^2} = 0,$$

where $T(x)$ is the tension and $\rho(x)$ is the density. We have already seen this in the hanging chain problem–there the tension is proportional to $x$ and the density is constant. Let's go back to the simpler problem of a uniform string fixed at the ends $x = 0$ and $x = \pi$. In this case the tension and the density are both constant: say $T(x) = T$ and $\rho(x) = \rho$. Then

$$u_{xx} - \frac{\rho}{T}u_{tt} = 0,\ 0 < x < \pi$$
$$u(0,t) = u(\pi,t) = 0, \text{ and}$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

From our vast knowledge of eigenvalue problems we know to let $u = \sum_{n=1}^{n}\alpha_n(t)\sin nx$, which gives us

$$\sum_{n=1}^{\infty}\left[-n^2\alpha_n(t) - \frac{\rho}{T}\alpha_n''(t)\right]\sin x = 0.$$

Thus

$$\alpha_n''(t) + n^2\frac{T}{\rho}\alpha_n = 0,$$

which has solutions

$$\alpha_n(t) = a_n\cos nvt + b_n\sin nvt, \text{ where}$$
$$v = \sqrt{\frac{T}{\rho}}.$$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty}(a_n\cos nvt + b_n\sin nvt)\sin nx.$$

From the initial conditions, we know

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx, \text{ and } b_n = \frac{2}{\pi n v}\int_0^{\pi} g(x)\sin nx dx.$$

Observe that the solution $u$ is periodic in $t$: $u(x,t) = u(x, t + 2\pi/v)$

**Example.** Suppose $v = 1$, $g(x) = 0$, and

$$f(x) = \begin{cases} x/2 & 0 \le x \le \pi/2 \\ -(x-\pi)/2 & 1/2 < x \le 1 \end{cases}$$

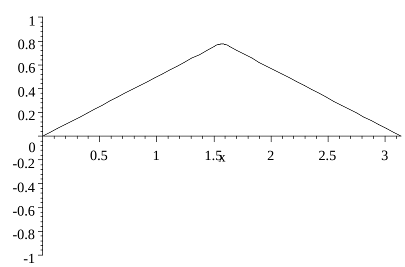
Then

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2\sin\frac{n\pi}{2}}{\pi n^2},$$

and $b_n = 0$. Hence,

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{2}}{n^2}\cos nt \sin nx.$$

Let's see what this looks like for a sequence of values of time $t$.




$t = 0$

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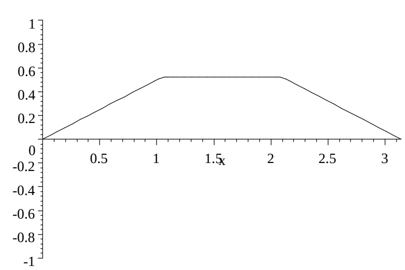
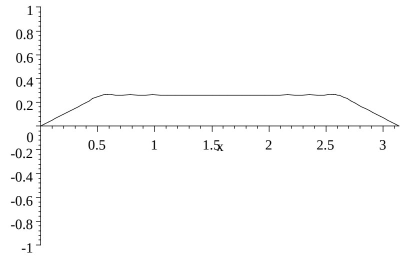
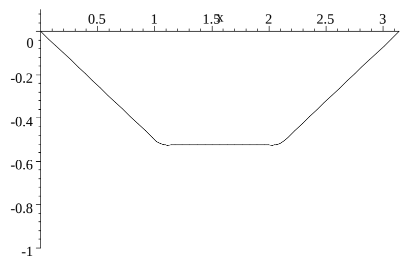


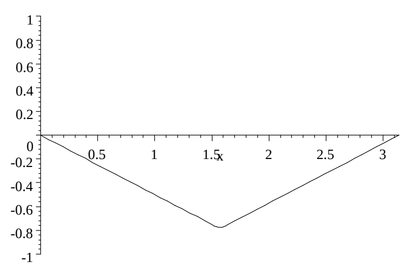

$t = \pi/6$




$t = \pi/3$




$t = 5\pi/6$

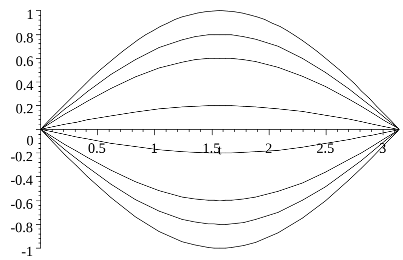



$t = \pi$

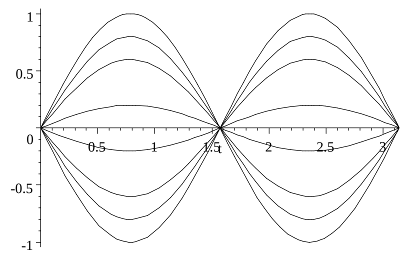
Suppose all the terms in the series for $u$ save the one for $n = 1$ are zero. The solution then is simply

$$\begin{aligned} u(x,t) &= (a_1 \cos \nu t + b_1 \sin \nu t) \sin x \\ &= A \cos(\nu t + \varphi) \sin x. \end{aligned}$$

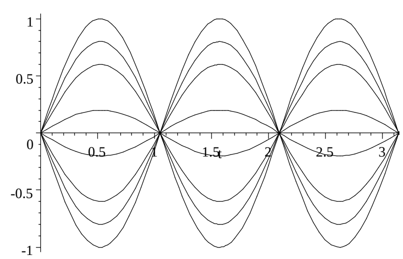
In this oscillation, the string always has the shape of a single arch of the sine curve and vibrates with the radian frequency $\nu$:




The solution for all $n = 0$ except $n = 2$ has the form $A\cos(2vt + \varphi)\sin 2x$. Here the string has the shape of $\sin 2x$ and oscillates with a frequency $2v$, or twice the frequency of the previous solution:




Musically, this oscillation would sound an octave higher than the first one. Convince yourself that for $n = 3$, the string would vibrate with frequency $3v$:




What is the musical interval between this one and the previous one?

The solutions like these in which all but one of values of $n$ are zero are called **vibration modes**. Thus every solution is a superposition, or ”sum”, of these modes of vibration.

Let’s consider again the solution in which $g(x) = 0$. Then

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos nvt \sin nx.$$

Drawing upon our vast knowledge of trigonometry, we see that

$$\cos nvt \sin nx = \frac{1}{2}[\sin(n(x+vt)) + \sin(n(x-vt))].$$

Hence,

$$\begin{aligned} u(x,y) &= \frac{1}{2}\sum_{n=1}^{\infty} a_n[\sin(n(x+vt)) + \sin(n(x-vt))] \\ &= \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n \sin(n(x+vt)) + \sum_{n=1}^{\infty} a_n \sin(n(x-vt))\right). \end{aligned}$$

Now we know that for most all $x$

$$\tilde{f}(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

where $\tilde{f}$ is the odd periodic extension of $f$. Thus

$$\tilde{f}(x+vt) = \sum_{n=1}^{\infty} a_n \sin(n(x+vt)), \text{ and } \tilde{f}(x-vt) = \sum_{n=1}^{\infty} a_n \sin(n(x-vt)),$$

and our solution is

$$u(x,t) = \frac{1}{2}\left(\tilde{f}(x+vt) + \tilde{f}(x-vt)\right)$$

This is special case of what is called **D'Alembert's** formula of the wave equation. In case $g$ is not zero, there is the full version of D'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left(\tilde{f}(x+vt) + \tilde{f}(x-vt)\right) + \frac{1}{2v}\int_{x-vt}^{x+vt} g(s)ds$$