Chapter Eight - Eigenvalues and Eigenfunctions Again

We turn our attention now to problems in which there are two ”space variables”; *e.g*,

$$\nabla^2 u - u_t = 0,$$

where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

We thus consider the two dimensional eigenvalue problem

$$\begin{aligned} \nabla^2 \varphi &= -\lambda^2 \varphi, \ (x,y) \in R \\ \varphi &= 0 \text{ on the boundary of the region } R. \end{aligned}$$

Let’s begin with a square region $0 \le x, y \le \pi$. Then as usual we assume

$$\varphi(x,y) = \sum_{n=1}^{\infty} \alpha_n(y) \sin nx,$$

and get

$$\nabla^2 \varphi = \sum_{n=1}^{\infty} [-n^2 \alpha_n(y) + \alpha_n''(y)] \sin nx = \sum_{n=1}^{\infty} -\lambda^2 \alpha_n(y) \sin nx, \text{ or}$$

$$\sum_{n=1}^{\infty} [-n^2 \alpha_n(y) + \alpha_n''(y) - \lambda^2 \alpha_n(y)] \sin nx = 0.$$

Thus we need

$$\begin{aligned} -n^2 \alpha_n(y) + \alpha_n''(y) + \lambda^2 \alpha_n(y) &= 0, \text{ or} \\ \alpha_n''(y) + (\lambda^2 - n^2)\alpha_n(y) &= 0. \end{aligned}$$

So,

$$\alpha_n(y) = a_n \cos \sqrt{\lambda^2 - n^2}\, y + b_n \sin \sqrt{\lambda^2 - n^2}\, y, \text{ where}$$

This gives us

$$\varphi(x,y) = \sum_{n=1}^{\infty}\left(a_n\cos\sqrt{\lambda^2-n^2}\,y + b_n\sin\sqrt{\lambda^2-n^2}\,y\right)\sin nx$$

The requirements that $\varphi(x,0) = 0$ and $\varphi(x,\pi) = 0$ become

$$\sum_{n=1}^{\infty} a_n \sin nx = 0, \text{ which means we must have } a_n = 0; \text{ and}$$

$$\sum_{n=1}^{\infty} b_n \sin\sqrt{\lambda^2-n^2}\,\pi \sin nx = 0.$$

This gives us an infinite collection of equations $b_n \sin\sqrt{\lambda^2-p^2}\,\pi = 0$, $n = 1,2,3,\ldots$.

Suppose we have a solution in which $b_n \neq 0$. Then it must be true that $\sin\sqrt{\lambda^2-n^2}\,\pi = 0$, or $\sqrt{\lambda^2-n^2} = m$, an integer $m = 0,1,2,\ldots$. That is, $m^2 = \lambda^2 - n^2$, or $\lambda_{nm}^2 = m^2 + n^2$. In other words, we have found eigenvalues $\lambda_{nm}^2 = m^2 + n^2$, with the corresponding eigenfunctions $\varphi_{nm}(x,y) = \sin my \sin nx$. It is not hard to convince yourself there are no other eigenvalues.

Example. The problem of finding the temperature $u(x,y,t)$ in a square now looks very similar to the one-dimensional problem.

$$\begin{aligned} u_{xx} - u_t &= 0,\ 0 < x,y < \pi,\ t > 0 \\ u(x,0) = u(x,\pi) &= u(0,y) = u(\pi,y) = 0, \text{ and} \\ u(x,y,0) &= f(x,y). \end{aligned}$$

We let $u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{nm}(t)\sin my \sin nx$. Hence,

$$u_{xx} - u_t = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[-\lambda_{nm}^2\alpha_{nm}(t) - \alpha'_{nm}(t)]\sin my \sin nx = 0.$$

And so,

$$-\lambda_{nm}^2\alpha_{nm}(t) - \alpha'_{nm}(t) = 0$$

gives us

$$\alpha_{nm}(t) = a_{nm}e^{-\lambda_{nm}t}.$$

Thus,

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}e^{-\lambda_{nm}t}\sin my \sin nx.$$

We get the coefficients a_{nm} from the initial condition:

$$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin my \sin nx = f(x,y).$$

$$\begin{aligned} a_{nm} &= \frac{\int_0^{\pi}\int_0^{\pi} f(x,y)\sin my \sin nx dxdy}{\int_0^{\pi}\int_0^{\pi}(\sin my \sin nx)^2 dxdy} \\ &= \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(x,y)\sin my \sin nx dxdy \end{aligned}$$

Next we turn our attention to the eigenvalue problem in case the region R is a disc of radius c centered at the origin. Using polar coordinates gives us

$$\begin{aligned} \nabla^2\varphi(r,\theta) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = -\lambda^2\varphi \\ \varphi(c,\theta) &= 0. \end{aligned}$$

In the usual way, this leads us to the one-dimensional eigenvalue problem

$$\begin{aligned} \xi'' + \mu^2\xi = 0,& \ -\pi < \theta < \pi \\ \xi(-\pi) = \xi(\pi)& \text{ and} \\ \xi'(-\pi) = \xi'(\pi)& \end{aligned}$$

We have seen that this problem has eigenvalues μ_n^2, where $\mu_n = 0,1,2,\ldots,$ with corresponding eigenfunctions $\xi_0 = 1$, $\xi_{1n} = \cos n\theta$, and $\xi_{2n} = \sin n\theta$. We thus set

$$u(r,\theta) = \alpha_0(r) + \sum_{n=1}^{\infty}[\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta],$$

which substituted into the original equation leads to

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\alpha_0}{\partial r}\right)+\lambda^2\alpha_0+\sum_{n=1}^{\infty}\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\alpha_n}{\partial r}\right)+\left(\lambda^2-\frac{n^2}{r^2}\right)\alpha_n\right]\cos n\theta\right.\\
&\left.+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\beta_n}{\partial r}\right)+\left(\lambda^2-\frac{n^2}{r^2}\right)\beta_n\right]\sin n\theta\right\}\\
&=0.
\end{aligned}
$$

Then we have the differential equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right)+\left(\lambda^2-\frac{n^2}{r^2}\right)y=0,\ n=0,1,2,3,\ldots$$

Or,

$$\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right)+\left(r\lambda^2-\frac{n^2}{r}\right)y=0.$$

This is, of course, the celebrated **Bessel's Equation.** We know that all solutions look like

$$y(r)=AJ_n(\lambda r)+BY_n(\lambda r),$$

where J_n is the *Bessel function of the first kind of order n* and Y_r is, not surprisingly, the *Bessel function of the second kind of order n.* The function Y_n is not nice at $r=0$, and so $B=0$. Hence $y=AJ_n(\lambda r)$.

Putting these solutions back in the original expression for $u(r,\theta)$ gives us

$$
\begin{aligned}
u(r,\theta)&=\alpha_0(r)+\sum_{n=1}^{\infty}[\alpha_n(r)\cos n\theta+\beta_n(r)\sin n\theta]\\
&=a_0J_0(\lambda r)+\sum_{n=1}^{\infty}[a_nJ_n(\lambda r)\cos n\theta+b_nJ_n(\lambda r)\sin n\theta].
\end{aligned}
$$

Now the requirement that $u(c,\theta)=0$ tells us that we must have

$$
\begin{aligned}
&a_0J_0(\lambda c)=0,\\
&a_nJ_n(\lambda c)=0,\text{ and}\\
&b_nJ_n(\lambda c)=0,\text{ for } n=1,2,3,\ldots
\end{aligned}
$$

Reflect on this system of equations. Clearly we can not have a nonzero eigenfunction u if all the coefficients a_n and b_n are zero. First, suppose $a_0\neq 0$. Then we must have

$$J_0(\lambda c) = 0.$$

In other words, λc must be a zero of J_0. There are infinitely many of these, call them z_{0m}, for $m = 1, 2, 3, \ldots$. Let $\lambda_{0m} = z_{0m}/c$, for $m = 1, 2, 3, \ldots$. Now the remainder of the equations become

$$a_n J_n(\lambda_{0m} c) = a_n J_n(z_{0m}) = 0 \text{ and } b_n J_n(z_{0m}) = 0, n = 1, 2, 3, \ldots.$$

We know that $J_n(z_{0m}) \neq 0$ and so it must be true that $a_n = b_n = 0$, for $n \geq 1$. Thus the eigenfunctions corresponding to λ_{0m} are $J_0(\lambda_{0m})$.

Next, suppose we have a solution in which $a_k \neq 0$ for some $k \geq 1$. Then we have the equation $a_k J_k(\lambda c) = 0$ and because $a_k \neq 0$, it must be true that $\lambda c = z_{km}$, where z_{km} is the m^{th} zero of $J_k(r)$. Let $\lambda_{km} = z_{km}/c$. Then we have $b_k J_k(\lambda_{km} c) = 0$ for any b_k, and $J_n(\lambda_{nm} c) \neq 0$ for $n \neq k$. It follows that all a_n and b_n for $n \neq k$ must be zero, and substituting back in our expression for $u(r,\theta)$, gives us two independent eigenfunctions corresponding to λ_{km}. They are $J_k(\lambda_{km} r)\cos k\theta$ and $J_k(\lambda_{km} r)\sin\theta$.

To summarize: there is a two dimensional array of eigenvalues, λ_{nm}^2, with $\lambda_{nm} = z_{nm}/c$, where z_{nm} is the m^{th} zero of J_n. The corresponding eigenfunctions are $J_0(\lambda_{0m} r)$, and $J_k(\lambda_{km} r)\cos k\theta$ and $J_k(\lambda_{km} r)\sin k\theta$ for $k \geq 1$.

Example. Consider

$$\begin{aligned} \nabla^2 u - u_t &= 0,\ 0 \leq r < a, -\pi < \theta \leq \pi, t > 0 \\ u(a,\theta,t) &= 0,\ \text{and} \\ u(r,\theta,0) &= g(r,\theta). \end{aligned}$$

It should be obvious why we let

$$u(r,\theta,t) = \sum_{m=1}^{\infty} \alpha_{0m}(t) J_0(\lambda_{0m} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [\alpha_{nm}(t)\cos n\theta + \beta_{nm}(t)\sin n\theta] J_n(\lambda_{nm} r).$$

It should be clear to one and all that

$$u(r,\theta,t) = \sum_{m=1}^{\infty} a_{0m} e^{-\lambda_{nm}^2 t} J_0(\lambda_{0m} r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [a_{nm}\cos n\theta + b_{nm}\sin n\theta] e^{-\lambda_{nm}^2 t} J_n(\lambda_{nm} r),$$

where

$$a_{0m} = \frac{\int_0^c \int_{-\pi}^{\pi} r g(r,\theta) J_0(\lambda_{0m} r)\, d\theta\, dr}{2\pi \int_0^c r (J_0(\lambda_{0m} r))^2\, dr}, \quad m = 1, 2, 3, \ldots$$

$$a_{nm} = \frac{\int_0^c \int_{-\pi}^{\pi} rg(r,\theta) J_n(\lambda_{nm} r) \cos n\theta \, d\theta dr}{\int_0^c \int_{-\pi}^{\pi} r(J_n(\lambda_{nm} r) \cos n\theta)^2 d\theta dr}, \; m,n = 1,2,3,\ldots$$

$$b_{nm} = \frac{\int_0^c \int_{-\pi}^{\pi} rg(r,\theta) J_n(\lambda_{nm} r) \sin n\theta \, d\theta dr}{\int_0^c \int_{-\pi}^{\pi} r(J_n(\lambda_{nm} r) \sin n\theta)^2 d\theta dr}, \; m,n = 1,2,3,\ldots$$