Chapter Nine - Vibrating Membranes

The vibrating membrane problem is simply the two-dimensional version of the vibrating string problems. Specifically, we are given a plane region R and we want to find $u(x,y,t)$ so that

$$\nabla^2 u - \frac{1}{k^2}u_{tt} = 0, \text{ for } (x,y) \in R,\ t > 0$$
$$u(x,y,t) = 0 \text{ for } (x,y) \in \text{ boundary of } R;$$
$$u(x,y,0) = f(x,y) \text{ and } u_t(x,y,0) = g(x,y).$$

Here k is a constant which depends on the physical properties of the membrane–density and tension.

We begin with the case in which R is a disc of radius c and centered at the origin. It should come as no surprise that we use polar coordinates. Fortunately, we know all about the associated eigenvalue problem

$$\nabla^2\varphi = -\lambda^2\varphi$$
$$\varphi(c,\theta) = 0.$$

Recall we have eigenvalues $\lambda_{nm} = z_{nm}/c$, $n = 0,1,2,\ldots$, $m = 1,2,3,\ldots$, where z_{nm} is the m^{th} zero of the Bessel function J_n. The corresponding eigenfunctions φ_{nm} are

$$\varphi_{0m}(r,\theta) = J_0(\lambda_{0m}r),\ m = 1,2,3,\ldots$$
$$\varphi_{nm}(r,\theta) = \begin{cases} J_n(\lambda_{nm}r)\cos n\theta \\ J_n(\lambda_{nm}r)\sin n\theta \end{cases}.$$

We thus let

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\alpha_{0m}(t)J_0(\lambda_{0m}r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\alpha_{nm}(t)\cos n\theta + \beta_{nm}(t)\sin n\theta]J_n(\lambda_{nm}r).$$

Hence,

$$
\begin{aligned}
\nabla^2 u - \frac{1}{k^2}u_{tt} \;=\; & \\
& \sum_{m=1}^{\infty}\left[-\lambda_{0m}^2\alpha_{0m}(t) - \frac{1}{k^2}\alpha_{0m}''(t)\right]J_0(\lambda_{0m}r) + \\
& \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\left[-\lambda_{nm}^2\alpha_{nm}(t) - \frac{1}{k^2}\alpha_{0m}''(t)\right]\cos n\theta + \right. \\
& \left.\left[-\lambda_{nm}^2\beta_{nm}(t) - \frac{1}{k^2}\beta_{0m}''(t)\right]\sin n\theta\right\}J_n(\lambda_{nm}r)
\end{aligned}
$$

We now have a bunch of ordinary differential equations of the form

$$\gamma''(t) + \lambda^2 k^2 \gamma(t) \;=\; 0.$$

Thus,

$$
\begin{aligned}
\alpha_{nm}(t) \;&=\; a_{nm}\cos\lambda_{nm}kt + b_{nm}\sin\lambda_{nm}kt. \\
\beta_{nm}(t) \;&=\; c_{nm}\cos\lambda_{nm}kt + d_{nm}\sin\lambda_{nm}kt,
\end{aligned}
$$

We substitute these vales back into our expression for $u(r,\theta,t)$ and when the dust settles, we see

$$
\begin{aligned}
u(r,\theta,t) \;=\; & \sum_{m=1}^{\infty}[a_{0m}\cos\lambda_{0m}kt + b_{0m}\sin\lambda_{0m}kt]J_0(\lambda_{0m}r) + \\
& \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{[a_{nm}\cos\lambda_{nm}kt + b_{nm}\sin\lambda_{nm}kt]J_n(\lambda_{nm}r)\cos n\theta + \\
& [c_{nm}\cos\lambda_{nm}kt + d_{nm}\sin\lambda_{nm}kt]J_n(\lambda_{nm}r)\sin n\theta\}
\end{aligned}
$$

All the constants are determined from the initial conditions. First, $u(r,\theta,0) \;=\; f(r,\theta)$ gives us

$$
\begin{aligned}
u(r,\theta,0) \;=\; f(r,\theta) \;=\; & \sum_{m=1}^{\infty}a_{0m}J_0(\lambda_{0m}r) + \\
& \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_{nm}J_n(\lambda_{nm}r)\cos n\theta + c_{nm}J_n(\lambda_{nm}r)\sin n\theta]
\end{aligned}
$$

Hence,

$$a_{0m} = \frac{\int_{-\pi}^{\pi}\int_{0}^{c} rf(r,\theta)J_0(\lambda_{0m}r)drd\theta}{\int_{-\pi}^{\pi}\int_{0}^{c} r(J_0(\lambda_{0m}r))^2 drd\theta},$$

$$a_{nm} = \frac{\int_{-\pi}^{\pi}\int_{0}^{c} rf(r,\theta)J_n(\lambda_{nm}r)\cos n\theta drd\theta}{\int_{-\pi}^{\pi}\int_{0}^{c} r(J_0(\lambda_{0m}r)\cos n\theta)^2 drd\theta}, \text{ and}$$

$$c_{nm} = \frac{\int_{-\pi}^{\pi}\int_{0}^{c} rf(r,\theta)J_n(\lambda_{nm}r)\sin n\theta drd\theta}{\int_{-\pi}^{\pi}\int_{0}^{c} r(J_n(\lambda_{nm}r)\sin n\theta)^2 drd\theta}.$$

Exercises

1. Find expressions for the constants b_{nm} and d_{nm} in the preceding discussion.

Consider a solution u_{0m} in which all series terms except those for λ_{0m} are zero. Thus,

$$\begin{aligned} u(r,\theta,t) &= [a_{0m}\cos\lambda_{0m}kt + b_{0m}\sin\lambda_{0m}kt]J_0(\lambda_{0m}r) \\ &= A\cos(\lambda_{0m}kt + \eta)J_0(\lambda_{0m}r). \end{aligned}$$

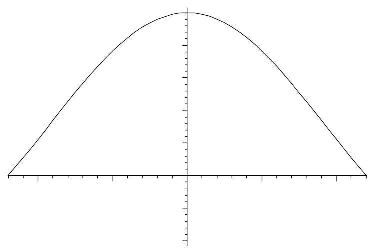
Ignoring the phase shift η, we have that u is a constant multiple of u_{0m} :

$$u_{0m} = \cos(\lambda_{0m}kt)J_0(\lambda_{0m}r).$$

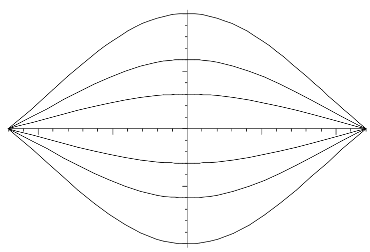
Let's see what this looks like. Notice first, that the solution does not depend on θ. Thus anywhere you take a cross-sectional slice of the membrane, you see the same curve. Initially, we see

$$u_{0m}(r,\theta,0) = J_0(\lambda_{0m}r).$$

For $m = 1$, we have $u_{01} = J_0(\lambda_{01}r)$. This looks like

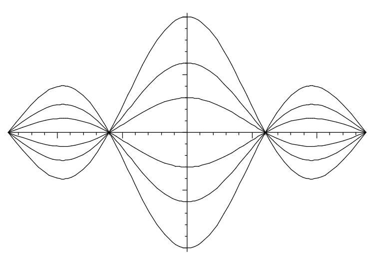


Now, as t increases, we see this same shape, but with an amplitude of $\cos(\lambda_{01}kt)$. Here is a picture for a few values of t:

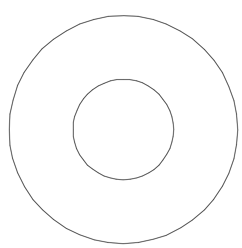


The membrane thus oscillates up and down with a frequency of $\lambda_{01}k = 2.405k$.

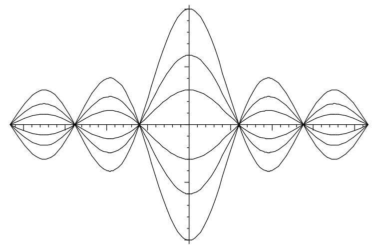
For $m = 2$, the membrane oscillates with a frequency of $\lambda_{02}k = 5.520k$, and the corresponding pictures of the cross-section look like

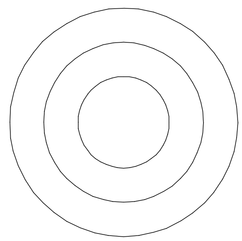


Observe that here there is a so-called nodal curve–a set of points that do not move. It is a circle of radius $(\lambda_{01}/\lambda_{02})c = 0.43569c$. Looking down on the membrane, we see



I hope it is clear that the corresponding pictures for an oscillation frequency $1/\lambda_{03}k$ look like





Exercises

2. Find the radii of the nodal circles for the solution $u_{03} = \cos(\lambda_{03}kt)J_0(\lambda_{03}r)$.

3. Describe and draw pictures of the oscillation having frequency $\lambda_{05}k$.

Next, consider the solution in which all the terms save the ones for λ_{11} are zero. Here

$$\begin{aligned} u(r,\theta,t) = {} & [a_{11}\cos\lambda_{11}kt + b_{11}\sin\lambda_{11}kt]J_1(\lambda_{11}r)\cos\theta + \\ & [c_{11}\cos\lambda_{11}kt + d_{11}\sin\lambda_{11}kt]J1(\lambda_{11}r)\sin\theta. \end{aligned}$$

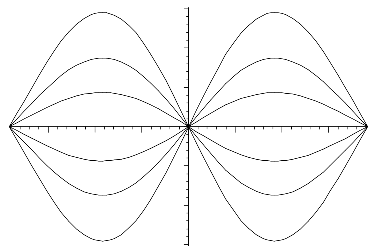
Look at the first of the two terms:

$$\begin{aligned} \tilde{u}(r,\theta,t) &= [a_{11}\cos\lambda_{11}kt + b_{11}\sin\lambda_{11}kt]J_1(\lambda_{11}r)\cos\theta \\ &= A\cos(\lambda_{11}kt + \eta)J_1(\lambda_{11}r)\cos\theta. \end{aligned}$$

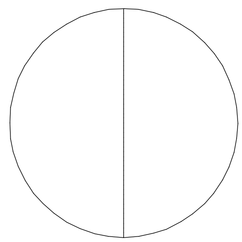
As before, look at

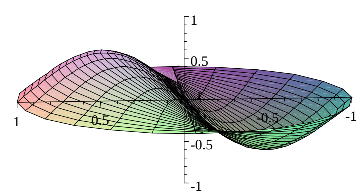
$$u_{11}(r,\theta,t) = \cos(\lambda_{11}kt)J_1(\lambda_{11}r)\cos\theta.$$

In this case, a cross-sectional slice through the membrane does indeed depend on θ. For $\theta = 0$, we see $\cos(\lambda_{11}kt)J_1(\lambda_{11}r)$ and for $\theta = \pi$, we see $-\cos(\lambda_{11}kt)J_1(\lambda_{11}r)$. Thus, for various values of t, this slice looks like



As we take slices for increasing values of θ, the picture looks the same, but with decreasing amplitude until $\theta = \pi/2$, at which the amplitude is zero; *i.e.*, there is a nodal line. Look at the picture.

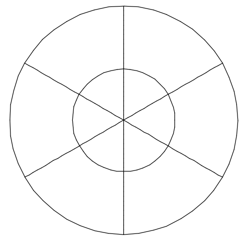




Note that we get nothing new from the $\cos(\lambda_{11}kt)J_1(\lambda_{11}r)\sin\theta$ term; it is just this picture turned ninety degrees. It should be clear how to see what the remaining vibration modes look like. For the solution corresponding to λ_{nm}, the nodal lines are the solutions to the equation

$$J_n(\lambda_{nm}r)\cos n\theta.$$

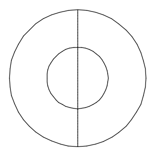
For instance, for $(n.m) = (3,2)$, the nodal lines look like

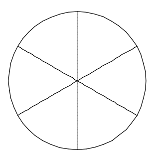


A **normal mode** is a solution in which each point of the membrane oscillates about equilibrium with the same frequency. Thus the solutions just discussed are normal modes.

Exercises

4. The picture shows the nodal lines of a vibrating membrane (same membrane). Which is vibrating with the higher frequency? Explain.





Now let's consider the case of a square membrane:

$$\nabla^2 u - \frac{1}{k^2}u_{tt} = 0,\ 0 < x,y < L,\ t > 0$$
$$u(x,0) = u(L,y) = u(x,L) = u(0,y) = 0, \text{ and}$$
$$u(x,y,0) = f(x,y),\ \ u_t(x,y,0) = g(x,y)..$$

From our previous work on eigenvalue problems, *etc.*, we know to let

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{nm}(t)\sin\frac{m\pi}{L}x\sin\frac{n\pi}{L}y,$$

which gives us the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_{nm}\cos\lambda_{nm}kt + b_{nm}\sin\lambda_{nm}kt]\sin\frac{m\pi}{L}x\sin\frac{n\pi}{L}y,$$

where

$$\lambda_{nm}^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2,$$

and the constants a_{nm} and b_{nm} are determined from the initial conditions.

Exercise

5. Find expressions for the constants a_{nm} and b_{nm}.

The normal modes of oscillation are more exciting in this case. Now if we assume the membrane vibrates with a fundamental frequency λ_{nm}, there is a significant complication compared to the situation with a circular membrane. In the circular case, the integers n and m determined exactly one term of our series as the solution. Here this is not the case. The normal modes are not simply the terms

$$\begin{aligned} u_{nm} &= (a_{nm}\cos\lambda_{nm}kt + b_{nm}\sin\lambda_{nm}kt)\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}y \\ &= A\cos(\lambda_{nm}kt + \eta)\sin\frac{m\pi}{L}x\sin\frac{n\pi}{L}y \end{aligned}$$

because the frequency $\lambda_{nm}k$ may appear in more that one such term. Let's illustrate with an example. Suppose $k = 1$, and $L = \pi$. Then $\lambda_{nm} = \sqrt{n^2 + m^2}$. Now look at the natural frequency $\lambda_{13} = \lambda_{31} = \sqrt{10}$. Then all solutions of the form

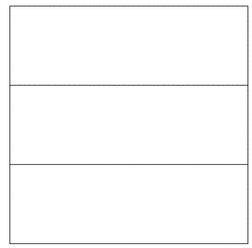
$$\cos(t\sqrt{10})(a\sin x\sin 3y + b\sin 3x\sin y)$$

are normal modes.

Look at the case where $b = 0$. Then each point (x,y) moves sinusoidally up and down with the frequency $\sqrt{10}$ with an amplitude given by a constant times

$$s(x,y) = \sin x\sin 3y$$

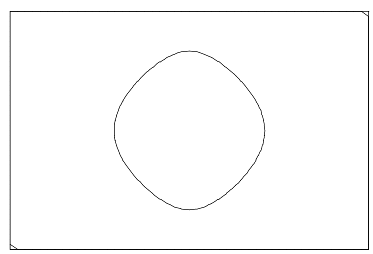
The nodal lines are simply places at which $s(x,y) = 0$.



But with this same frequency, we also have lots of other normal modes. Let's take a look at one of them:

$$\cos(t\sqrt{10})(\sin x \sin 3y + 2\sin 3x \sin y)$$

The nodal lines:



As you can imagine, the possibilities are almost endless.

Exercises

6. Draw some more graphs of nodal lines for normal modes of frequecy $\sqrt{10}$.