March 23

Consider the eigenvalue problem on the disc of radius c centered at the origin:

$$\nabla^2 u = -\mu^2 u$$
$$u = 0 \text{ on the boundary of the disc}$$

In polar coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\mu^2 u.$$

In the usual way, this leads us to the eigenvalue problem

$$\varphi'' + \lambda^2\varphi = 0,\ -\pi < \theta < \pi$$
$$\varphi(-\pi) = \varphi(\pi) \text{ and}$$
$$\varphi'(-\pi) = \varphi'(\pi)$$

We have seen that this problem has eigenvalues λ_n^2, where $\lambda_n = 0, 1, 2, \ldots$, with corresponding eigenfunctions $\varphi_0 = 1$, $\varphi_{1n} = \cos n\theta$, and $\varphi_{2n} = \sin n\theta$. We thus set

$$u(r,\theta) = \alpha_0(r) + \sum_{n=1}^{\infty}[\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta],$$

which substituted into the original equation leads to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \alpha_0}{\partial r}\right) + \mu^2\alpha_0 + \sum_{n=1}^{\infty}\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \alpha_n}{\partial r}\right) + \left(\mu^2 - \frac{n^2}{r^2}\right)\alpha_n\right]\cos n\theta\right.$$
$$\left.+\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \beta_n}{\partial r}\right) + \left(\mu^2 - \frac{n^2}{r^2}\right)\beta_n\right]\sin n\theta\right\}$$
$$= 0.$$

Then we have the differential equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right) + \left(\mu^2 - \frac{n^2}{r^2}\right)y = 0,\ n = 0, 1, 2, 3, \ldots$$

Or,

$$\frac{\partial}{\partial r}\left(r\frac{\partial y}{\partial r}\right) + \left(r\mu^2 - \frac{n^2}{r}\right)y = 0.$$

This is, of course, the celebrated **Bessel's Equation.** We know that all solutions look like

$$y(r) = AJ_n(\mu r) + BY_n(\mu r),$$

where J_n is the *Bessel function of the first kind of order n* and Y_r is, not surprisingly, the *Bessel function of the second kind of order n.* The function Y_n is not nice at $r = 0$, and so $B = 0$. Hence $y = AJ_n(\mu r)$.

Putting these solutions back in the original expression for $u(r,\theta)$ gives us

$$\begin{aligned} u(r,\theta) &= \alpha_0(r) + \sum_{n=1}^{\infty}[\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta] \\ &= a_0 J_0(\mu r) + \sum_{n=1}^{\infty}[a_n J_n(\mu r)\cos n\theta + b_n J_n(\mu r)\sin n\theta]. \end{aligned}$$

Now the requirement that $u(c,\theta) = 0$ tells us that we must have

$$\begin{aligned} &a_0 J_0(\mu c) = 0, \\ &a_n J_n(\mu c) = 0, \text{ and} \\ &b_n J_n(\mu c) = 0, \text{ for } n = 1,2,3,\ldots \end{aligned}$$

Reflect on this system of equations. Clearly we can not have a nonzero eigenfunction u if all the coefficients a_n and b_n are zero. First, suppose $a_0 \neq 0$. Then we must have

$$J_0(\mu c) = 0.$$

In other words, μc must be a zero of J_0. There are infinitely many of these, call them z_{0m}, for $m = 1,2,3,\ldots$. Let $\mu_{0m} = z_{0m}/c$, for $m = 1,2,3,\ldots$. Now the remainder of the equations become $a_n J_n(\mu_{0m}c) = a_n J_n(z_{0m}) = 0$ and $b_n J_n(z_{0m}) = 0, n = 1,2,3,\ldots$. We know that $J_n(z_{0m}) \neq 0$ and so it must be true that $a_n = b_n = 0$, for $n \geq 1$. Thus the eigenfunctions corresponding to μ_{0m} are $J_0(\mu_{0m})$.

Next, suppose we have a solution in which $a_k \neq 0$ for some $k \geq 1$. Then we have the equation $a_k J_k(\mu c) = 0$ and because $a_k \neq 0$, it must be true that $\mu c = z_{km}$, where z_{km} is the m^{th} zero of $J_k(r)$. Let $\mu_{km} = z_{km}/c$. Then we have $b_k J_k(\mu_{km}c) = 0$ for any b_k, and $J_n(\mu_{nm}c) \neq 0$ for $n \neq k$. It follows that all a_n and b_n for $n \neq k$ must be zero, and substituting back in our expression for $u(r,\theta)$, gives us two independent eigenfunctions corresponding to μ_{km}. They are $J_k(\mu_{km}r)\cos k\theta$ and $J_k(\mu_{km}r)\sin\theta$.

To summarize: there is a two dimensional array of eigenvalues, μ_{nm}^2, with $\mu_{nm} = z_{nm}/c$, where z_{nm} is the m^{th} zero of J_n. The corresponding eigenfunctions are $J_0(\mu_{0m}r)$, and $J_k(\mu_{km}r)\cos k\theta$ and $J_k(\mu_{km}r)\sin k\theta$ for $k \geq 1$.