## Laplace's Equation on a Disc

Let's look at Laplace's equation

$$\nabla^2 u = 0$$

on the disc of radius $a$ and centered at the origin. Specifically, consider the problem

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x^2 + y^2 \le c^2, \\ u &= f \text{ on the boundary } x^2 + y^2 = c^2. \end{aligned}$$

In polar coordinates, the Laplacian operator looks like

$$\nabla^2 u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Thus we have

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0. \\ u(c,\theta) &= g(\theta). \end{aligned}$$

I hope it is clear from all that has gone before that we should consider the eigenvalue problem

$$\begin{aligned} \frac{d^2\varphi}{d\theta^2} &= -\lambda^2\varphi \\ \varphi(\pi) &= \varphi(-\pi), \text{ and} \\ \varphi'(\pi) &= \varphi'(-\pi) \end{aligned}$$

From our vast knowledge of Sturm-Liouville problems, we know what to expect. Let's see what we get.

$$\varphi(\theta) = A\cos\lambda\theta + B\sin\lambda\theta$$

and so our boundary conditions become

$$\begin{aligned} A\cos\lambda\pi + B\sin\lambda\pi &= A\cos(-\lambda\pi) + B\sin(-\lambda\pi), \text{ and} \\ \lambda[-A\sin\lambda\pi + B\cos\lambda\pi] &= \lambda[A\sin(-\lambda\pi) - B\cos(-\lambda\pi)]. \end{aligned}$$

Or,

$$2B\sin\lambda\pi = 0$$
$$\lambda A\sin\lambda\pi = 0$$

A moment's reflection should convince you that we obtain eigenvalues $\lambda_n^2 = n$ for $n = 0, 1, 2\ldots$. Corresponding to the eigenvalue $\lambda_0^2 = 0$, we have the eigenfunction $\varphi(\theta) = 1$, and corresponding to each eigenvalue $\lambda_n^2 = n$, we have two independent eigenfunctions $\cos n\theta$ and $\sin n\theta$.

With

$$u(r,\theta) = \alpha_0(r) + \sum_{n=1}^{\infty}[\alpha_n(r)\cos n\theta + \beta_n(r)\sin n\theta]$$

we have

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \\
&= \frac{1}{r}\frac{d}{dr}(\alpha_0'(r)) + \sum_{n=1}^{\infty}[\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r))\cos n\theta + \frac{1}{r}\frac{d}{dr}(r\beta_n'(r))\sin n\theta \\
&\quad - n^2\frac{\alpha_n(r)}{r^2}\cos n\theta - n^2\frac{\beta_n(r)}{r^2}\sin n\theta] \\
&= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dr}(\alpha_0'(r)) \\
&+ \sum_{n=1}^{\infty}[\left(\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r)) - n^2\frac{\alpha_n(r)}{r^2}\right)\cos n\theta + \left(\frac{1}{r}\frac{d}{dr}(rr\beta_n'(r)) - n^2\frac{\beta_n(r)}{r^2}\right)\sin n\theta \\
&= 0.
\end{aligned}$$

This gives us the differential equations

$$\frac{1}{r}\frac{d}{dr}(r\alpha_0'(r)) = 0,$$

$$\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r)) - \frac{1}{r^2}n^2\alpha_n(r) = 0, \text{ and}$$

$$\frac{1}{r}\frac{d}{dr}(r\beta_n'(r)) - \frac{1}{r^2}n^2\beta_n(r) = 0.$$

The first one is easy: $r\alpha_0'(r) = A$. Thus, $\alpha_0(r) = A\log r + B$. The requirement that the solution be nice at $r = 0$ means that $A$ must be 0. Thus $\alpha_0$ =constant $= a_0$. Next,

$$\frac{1}{r}\frac{d}{dr}(r\alpha_n'(r)) - \frac{1}{r^2}n^2\alpha_n(r) = 0 \text{ becomes}$$
$$r^2\alpha_n''(r) + r\alpha_n'(r) - n^2\alpha_n(r) = 0.$$

This, as you no doubt remember from Mrs. Turner's calculus class, is a so-called Cauchy-Euler equation, all solutions of which are

$$\alpha_n(r) = Ar^n + Br^{-n}.$$

Again, the solutions must be nice at $r = 0$, and so $B = 0$, and our solutions are

$$\alpha_n(r) = a_n r^n.$$

In exactly the same way, we get

$$\beta_n(r) = b_n r^n.$$

Putting it all together gives us

$$u(r,\theta) == a_0 + \sum_{n=1}^{\infty}[a_n r^n \cos n\theta + b_n r^n \sin n\theta].$$

The condition $u(c,\theta) = g(\theta)$ becomes

$$g(\theta) = a_0 + \sum_{n=1}^{\infty}[a_n c^n \cos n\theta + b_n c^n \sin n\theta].$$

Thus,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta,$$

$$a_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} g(\theta)\cos n\theta d\theta, \text{ and}$$

$$b_n = \frac{1}{\pi c^n}\int_{-\pi}^{\pi} g(\theta)\sin n\theta d\theta$$