Example

Consider the problem

$$\begin{aligned} u_{xx} - u_t &= 0,\ 0 \le x \le \pi,\ t > 0 \\ u(0,t) &= 0,\ u(\pi,t) = \sin t \\ u(x,0) &= 0 \end{aligned}$$

We need a problem in which we have homogeneous boundary conditions, so let $w(x,t) = u(x,t) - \frac{x}{\pi}\sin t$. Then we have

$$\begin{aligned} w_{xx} - w_t &= u_{xx} - u_t + \frac{x}{\pi}\cos t = \frac{x}{\pi}\cos t \\ w(0,t) &= u(0,t) = 0, \text{ and } w(\pi,t) = u(\pi,t) - \sin t = \sin t - \sin t = 0. \\ w(x,0) &= u(x,0) = 0 \end{aligned}$$

As usual, we look for a solution $w(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin nx$:

$$w_{xx} - w_t = \sum_{n=1}^{\infty} \{-n^2\alpha_n(t) - \alpha_n'(t)\} \sin nx = \frac{x}{\pi}\cos t$$

Next, I hope it is clear why we need the Fouier sine series for x.

$$x = \sum_{n=1}^{\infty} b_n \sin nx,$$

$$\text{where } b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx = \frac{2}{\pi}\left(\frac{\pi(-1)^{n+1}}{n}\right) = 2\frac{(-1)^{n+1}}{n}$$

Thus,

$$\sum_{n=1}^{\infty} \{-n^2\alpha_n(t) - \alpha_n'(t)\} \sin nx = \frac{x}{\pi}\cos t = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{\pi n}\cos t \sin nx,$$

Or, making one big series:

$$\sum_{n=1}^{\infty} \left\{-n^2\alpha_n(t) - \alpha_n'(t) + 2\frac{(-1)^n}{\pi n}\cos t\right\} \sin nx = 0.$$

Now we must cope with the differential equation

$$-n^2\alpha_n(t) - \alpha_n'(t) + 2\frac{(-1)^n}{\pi n}\cos t = 0, \text{ or}$$

$$\alpha_n'(t) + n^2\alpha_n(t) = 2\frac{(-1)^n}{\pi n}\cos t.$$

To solve this, mulitply by the integrating factor $e^{n^2 t}$:

$$e^{n^2 t}[\alpha_n'(t) + n^2\alpha_n(t)] = 2\frac{(-1)^n}{\pi n}e^{n^2 t}\cos t, \text{ or}$$

$$\frac{d}{dt}\left(e^{n^2 t}\alpha_{n(t)}\right) = 2\frac{(-1)^n}{\pi n}e^{n^2 t}\cos t.$$

Thus,

$$e^{n^2 t}\alpha_{n(t)} = 2\frac{(-1)^n}{\pi n}\left(\frac{n^2}{n^4+1}e^{n^2 t}\cos t + \frac{1}{n^4+1}e^{n^2 t}\sin t\right) + A_n,$$

and so

$$\alpha_n(t) = 2\frac{(-1)^n}{\pi n(n^4+1)}(n^2\cos t + \sin t) + A_n e^{-n^2 t}.$$

We're almost there:

$$\begin{aligned} w(x,t) &= \sum_{n=1}^{\infty}\alpha_n(t)\sin nx \\ &= \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{\pi n(n^4+1)}(n^2\cos t + \sin t) + A_n e^{-n^2 t}\right)\sin nx \end{aligned}$$

Finally, the initial condition:

$$w(x,0) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{\pi n(n^4+1)}(n^2) + A_n\right)\sin nx = 0$$

Hence,

$$A_n = -\frac{2n^2(-1)^n}{\pi n(n^4+1)},$$

and the whole gory mess is

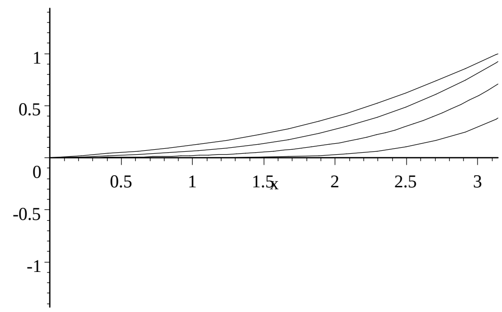
$$w(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^4+1)}\left(n^2(\cos t - e^{-n^2 t}) + \sin t\right)\sin nx.$$

At last!

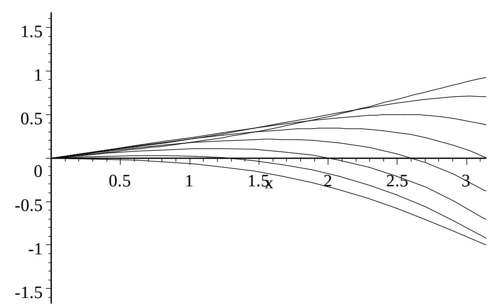
$$u(x,t) = w(x,t) + \frac{x}{\pi}\sin t, \text{ or}$$

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n^4+1)}\left(n^2(\cos t - e^{-n^2 t}) + \sin t\right)\sin nx + \frac{x}{\pi}\sin t$$

Let's take a look at some pictures. First, let's plot $u(x,t)$ for a sequence of values of t between 0 and $\pi/2$:



Take a look now at some pictures for t between $\pi/2$ and $3\pi/2$:
$u(x, 12\pi/8)$



Oooh, aah....