Math 2401 **Quiz Three** **February 22, 2001**

You may use any books, notes, tables, calculators, or computers you wish. Write your answers so that someone other than yourself can understand your exposition. Please do not return this sheet.

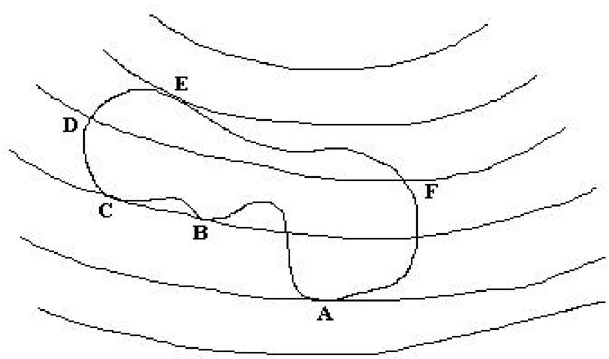
1. In the picture below, the closed curve is the graph of $g(x,y) = 0$, while the other curves are level curves of the function f. It is also true that $\nabla f \neq 0$ for all (x,y).

a)At which of the points **A, B, C, D, E, F** does a local extremum of f on $g(x,y) = 0$ occur? Explain!

A local extremum occurs at each of the points **A, B, C,** and **E**. At each point the level curve and the constraint curve $g(x,y) = 0$ are tangent, and the constraint curve does not cross the level curve–thus on $g(x,y) = 0$ the function f changes from decreasing to increasing, or from increasing to decreasing (We can't which because the level curves are not labeled.).

b)At which of the points **A, B, C, D, E, F** does an extremum of f on $g(x,y) = 0$ occur? Explain!

An extreme value occurs at **A** and **E**. Here the constraint curve stays completely on one side of the level curve to which it is tangent.



2. Find the point(s) on the curve $y = x^2$ closest to the point $(0,p)$, $p > 0$.
[Remember, the answer depends on p.]

Let's use the method of Lagrange multipliers. We seek the points at which the maximum value of $f(x,y) = x^2 + (y-p)^2$ occurs subject to the constraint $g(x,y) = y - x^2 = 0$. We need to solve the system

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = 0$$

Or,

$$\begin{aligned} 2x &= -\lambda 2x \\ 2(y-p) &= \lambda \\ y - x^2 &= 0 \end{aligned}$$

The first equation becomes $x(1+\lambda) = 0$. This has solutions $x = 0$ or $\lambda = -1$. If $x = 0$, then the third equation tells us that $y = 0$, also. Thus the point $(0,0)$ is a candidate for the closest point. If, on the other hand, $\lambda = -1$, then the second equation becomes $2(y-p) = -1$, or $y = p - 1/2$. We must have $x^2 = y$, so there are no solutions if $p < 1/2$. If $p \geq 1/2$, then $x = \sqrt{p-1/2}$, or $x = -\sqrt{p-1/2}$. To summarize, if $p < 1/2$, then there is only one candidate, $(0,0)$, and the minimum of f must occur there. In other words, when $p < 1/2$, the point closest to $(0,p)$ is $(0,0)$.

Now, when $p \geq 1/2$, we have three possibilities: $(0,0)$, $(\sqrt{p-1/2}, p-1/2)$, and $(-\sqrt{p-1/2}, p-1/2)$. Let's compute f at each.

$$\begin{aligned} f(0,0) &= p^2 \\ f(\sqrt{p-1/2}, p-1/2) &= p - 1/2 + 1/4 = p - 1/4, \text{ and} \\ f(-\sqrt{p-1/2}, p-1/2) &= p - 1/2 + 1/4 = p - 1/4. \end{aligned}$$

Now, which is smallest? Well, $p^2 - (p-1/4) = p^2 - p + 1/4 = (p-1/2)^2 \geq 0$, and so $p^2 \geq p - 1/2$. The minimum thus occurs at $(\sqrt{p-1/2}, p-1/2)$, and $(-\sqrt{p-1/2}, p-1/2)$. In other words, when $p \geq 1/2$, these are the points closest to $(0,p)$.

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