Iteration, Eigenvalues, and What All

1. Preliminaries. Suppose J is a Jordan block of order r:

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0\ldots & 0 \\ 0 & \lambda & 1 & 0\ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & \lambda & 1 \\ 0 & 0 & 0 & \ldots 0 & \lambda \end{bmatrix}$$

We can write this as

$$J = \lambda I + U,$$

where

$$U = \begin{bmatrix} 0 & 1 & 0 & 0\ldots & 0 \\ 0 & 0 & 1 & 0\ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ldots 0 & 0 \end{bmatrix}.$$

Thus,

$$J^k = (\lambda I + U)^k = \sum_{m=0}^{k} \lambda^{k-m} \binom{k}{m} U^m.$$

Notice that for $m \geq r$, we have $U^m = 0$. Hence for $k \geq r$,

$$J^k = \sum_{m=0}^{r} \lambda^{k-m} \binom{k}{m} U^m.$$

Now suppose that $|\lambda| < 1$, and observe that $\lim_{k \to \infty} \binom{k}{m} \lambda^{k-m} = 0$. Thus,

$$\lim_{k \to \infty} J^k = 0.$$

Proposition 1. Suppose that A is an $n \times n$ matrix such that $|\lambda| < 1$ for every eigenvalue λ. Then

$$\lim_{k \to \infty} A^k = 0.$$

Proof. This is easy. We know that A is similar to a matrix

$$S = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_p \end{bmatrix}$$

in which each J_q is a Jordan block. Thus

$$S^k = \begin{bmatrix} J_1^k & 0 & \dots & 0 \\ 0 & J_2^k & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \dots & J_p^k \end{bmatrix},$$

and we have just seen that each $J_q^k \to 0$ as $k \to \infty$.

2. The Neumann Series and Jacobi Iteration.

Proposition 2. Suppose that A is an $n \times n$ matrix such that $|\lambda| < 1$ for every eigenvalue λ. Then $I - A$ is invertible, and

$$(I - A)^{-1} = \lim_{k \to \infty} (I + A + A^2 + \dots + A^k).$$

Proof. This is a pretty easy consequence of the previous proposition. Let

$$S_k = I + A + A^2 + \dots + A^k.$$

Then,

$$AS_k = A + A^2 + \dots + A^{k+1},$$

and subtracting this from the first equation gives us

$$S_k - AS_k = S_k(I - A) = I - A^{k+1}.$$

Now simply looking at the limit of this as $k \to \infty$ gives us the conclusion of our proposition.
The series $\lim_{k \to \infty} (I + A + A^2 + \dots + A^k) = (I - A)^{-1}$ is called the **Neumann Series**.

Proposition 3. Suppose that A is an $n \times n$ matrix such that $|\lambda| < 1$ for every eigenvalue λ. For any initial guess x_0, the sequence (x_k) defined by

$$x_{k+1} = Ax_k + c$$

converges to the unique solution of the linear system $x = Ax + c$.

Proof. Again, we have a relatively easy consequence of our previous results. Simply observe that

$$x_{k+1} = Ax_k + c = A(Ax_{k-1} + c) + c = A^2 x_{k-1} + Ac + c$$

$$= A^2(Ax_{k-2} + c) + Ac + c = A^3x_{k-2} + A^2c + Ac + c$$
$$\vdots$$
$$= A^{k+1}x_0 + (A^k + A^{k-1} + \ldots + A + I)c.$$

We know from Proposition 1 that $A^{k+1} \to 0$ as $k \to \infty$ and from Proposition 2, that $A^k + A^{k-1} + \ldots + A + I \to (I - A)^{-1}$ as $k \to \infty$. Thus

$$x_{k+1} \to u = (I - A)^{-1}c, \text{ as } k \to \infty.$$

This iterative scheme is known as **Jacobi Iteration**.

3. **Gerschgorin discs**. Suppose A is a square matrix and $Av = \lambda v$. (In other words, v is an eigenvector of A and λ is the corresponding eigenvalue.). Let

$$|v_i| = \max\{|v_1|, |v_2|, \ldots, |v_n|\},$$

where $v = (v_1, v_2, \ldots, v_n)$. Now, the i^{th} component of Av is $\sum_{j=1}^{n} a_{ij}v_j$, and so we have

$$\sum_{j=1}^{n} a_{ij}v_j = \lambda v_i.$$

Thus,

$$\lambda v_i - a_{ii}v_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}v_j.$$

Then,

$$|\lambda - a_{ii}||v_i| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}||v_j| \leq |v_i| \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|,$$

or finally,

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| = r_i.$$

This shows that every eigenvalue of A lies in one of the discs centered at a_{ii} and having radius $r_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$. These discs are called **Gerschgorin discs**. These discs provide a way to estimate the magnitude of the eigenvalues of a matrix. We, for instance, the following

Proposition 4. Suppose $A = (a_{ij})$. Let $M = \max\{\sum_{j=1}^{n} |a_{1j}|, \sum_{j=1}^{n} |a_{2j}|, \ldots, \sum_{j=1}^{n} |a_{nj}|\}$. Then $|\lambda| \leq M$ for every eigenvalue λ of A.

Proof. Let λ be an eigenvalue of A. Then we know from the previous discussion that

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

for some i. But

$$|\lambda| - |a_{ii}| \leq |\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|.$$

Hence,

$$|\lambda| \leq \sum_{j=1}^{n} |a_{ij}| \leq M$$