Proof of Theorem 5.7

Theorem 5.7. Given an $n \times n$ matrix $A = (A_1, A_2, \ldots, A_n)$ and given any n-dimensional vector V, let B be the $n \times n$ matrix obtained from A by replacing the k^{th} A_k row by V, and let C be the matrix obtained by replacing k^{th} row A_k by $A_k + V$. Then

$$d(C) = d(A) + d(B).$$

Proof: Define the function f of the rows of A by

$$\begin{aligned} f(A_1, A_2, \ldots, A_n) &= d(C) - d(B) \\ &= d(A_1, \ldots, A_k + V, \ldots, A_n) - d(A_1, \ldots, V, \ldots, A_n). \end{aligned}$$

We prove the theorem by showing that $f(A_1, A_2, \ldots, A_n) = d(A)$.

First, suppose the set $\{A_1, A_2, \ldots, A_n\}$ is dependent. Then there are scalars c_j not all zero so $c_1A_1 + c_2A_2 + \ldots + c_nA_n = 0$. If $c_k = 0$, then $\{A_1, \ldots, A_{k-1}, A_{k+1}, \ldots, A_n\}$ is also dependent. In this case, we have $d(A) = d(B) = d(C) = 0$, and so $f(A) = 0$. Hence, $f(A) = d(A)$. Suppose, on the other hand, that $c_k \neq 0$. Then A_k is a linear combination of the other rows, and we have

$$d(A_1, \ldots, A_k + V, \ldots, A_n) = d(A_1, \ldots, V, \ldots, A_n).$$

Hence $f(A) = 0$, and, of course, $d(A) = 0$. Again, we see that $f(A) = d(A)$.

Next, suppose $\{A_1, A_2, \ldots, A_n\}$ is independent. Then this collection spans the space of all $n - tuples$, and so $V = \sum_{j=1}^{n} \alpha_j A_j$. We write $V = \alpha_k A_k + \sum_{j \neq k}^{n} \alpha_j A_j$. Then

$$\begin{aligned} d(A_1, \ldots, A_k + V, \ldots, A_n) &= d(A_1, \ldots, A_k + \alpha_k A_k + \sum_{j \neq k}^{n} \alpha_j A_j, \ldots, A_n) \\ &= d(A_1, \ldots, (1 + \alpha_k) A_k, \ldots, A_n). \end{aligned}$$

Also,

$$\begin{aligned} d(A_1, \ldots, V, \ldots, A_n) &= d(A_1, \ldots, \alpha_k A_k + \sum_{j \neq k}^{n} \alpha_j A_j, \ldots, A_n) \\ &= d(A_1, \ldots, \alpha_k A_k, \ldots, A_n). \end{aligned}$$

It follows that

$$\begin{aligned} f(A) &= d(A_1, \ldots, (1 + \alpha_k) A_k, \ldots, A_n) - d(A_1, \ldots, \alpha_k A_k, \ldots, A_n) \\ &= (1 + \alpha_k) d(A) - \alpha_k d(A) = d(A), \end{aligned}$$

and we are done.

Many thanks go to **Mr. Kapp** and **Mr. Roper**. The ideas for this proof are theirs; only the organization and exposition are mine.