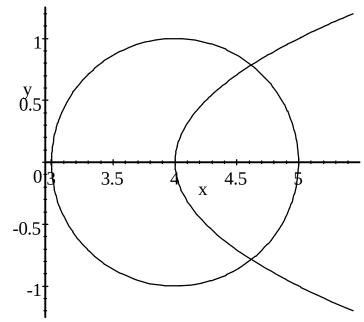
Math 2507B **Final Examination** **Spring 1999**

1. a)Give an iterated integral for the integral $\iint_R (xy + y^2)dA$, where R is the smaller of the two regions cut from the inside of the circle $(x-4)^2 + y^2 = 1$ by the parabola $y^2 = x - 4$.



We need to find where the curves intersect:

$$(x-4)^2 + y^2 = (y^2)^2 + y^2 = 1, \text{ or}$$

$$(y^2)^2 + y^2 - 1 = 0.$$

The quadratic formula gives us

$$y^2 = \frac{-1 \pm \sqrt{1+4}}{2}.$$

Clearly the negative solution makes no sense (y^2 isn't negative!), so

$$y = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}.$$

Now, for the integral:

$$\iint_R (xy + y^2)dA = \int_{-\sqrt{\frac{-1+\sqrt{5}}{2}}}^{\sqrt{\frac{-1+\sqrt{5}}{2}}} \int_{y^2+4}^{4+\sqrt{1-y^2}} (xy + y^2)dxdy$$

b)Give a **Maple** command for evaluating this:

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int(int(x*y+y^2,x=y^2+4..4+sqrt(1-y^2)),
x=-sqrt((-1+sqrt(5))/2)..sqrt((-1+sqrt(5))/2));
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2. Find the mass of a wire in the shape of the helix

$$\mathbf{r}(t) = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k},\ 0 \le t \le 2\pi$$

if the density at any point is the square of the distance to the origin.

$$\text{Mass} = \int_H (x^2+y^2+z^2)\,ds = \int_0^{2\pi} (t^2+\cos^2 t+\sin^2 t)|\mathbf{r}'(t)|\,dt.$$

Now, $\mathbf{r}'(t) = \mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$, and so $|\mathbf{r}'(t)| = \sqrt{1+\cos^2 t+\sin^2 t} = \sqrt{2}$. Thus,

$$\begin{aligned}\text{Mass} &= \int_0^{2\pi} (t^2+\cos^2 t+\sin^2 t)\sqrt{2}\,dt \\ &= \sqrt{2}\int_0^{2\pi}(t^2+1)\,dt \\ &= \sqrt{2}\left[\frac{8}{3}\pi^3+2\pi\right]\end{aligned}$$

3. a)Find a potential function for

$$\mathbf{F} = (y+2xyz+1)\mathbf{i} + (x^2z+x+z+1)\mathbf{j} + (x^2y+y+1)\mathbf{k}$$

If $\nabla g = \mathbf{F}$, then

$$\frac{\partial g}{\partial x} = y+2xyz+1, \text{ or}$$

$$g = xy + x^2yz + x + h(y,z)$$

To find $h(y,z)$, differentiate with respect to y:

$$\frac{\partial g}{\partial y} = x + x^2z + \frac{\partial h(y,z)}{\partial y} = x^2z+x+z+1.$$

Thus,

$$\frac{\partial h(y,z)}{\partial y} = z+1, \text{ and so}$$

$$h(y,z) = yz + y + k(z).$$

Hence,

$$g = xy + x^2yz + x + h(y,z) = xy + x^2yz + x + yz + y + k(z).$$

To find k, differentiate with respect to z :

$$\frac{\partial g}{\partial z} = x^2y + y + k'(z) = x^2y+y+1.$$

This gives us

$$k'(z) = 1, \text{ or}$$

$$k(z) = z.$$

Substituting back yields

$$\begin{aligned}g &= xy + x^2yz + x + yz + y + k(z) \\ &= xy + x^2yz + x + yz + y + z.\end{aligned}$$

b)Evaluate the integral $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the line segment from the origin to (1,2,3).

This is now easy:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = g(1,2,3) - g(0,0,0)$$
$$= 20 - 0 = 20$$

5. Find the area of that part of the plane that lies inside the cylinder $x^2 + y^2 = 4$.

Area $= \iint_P dA$. To find this integral, we need a vector description of our surface P. This is easy; simply take $s = x$ and $t = y$. This gives us

$$\mathbf{r}(s,t) = s\mathbf{i} + t\mathbf{j} + (4 - s - 2t)\mathbf{k},$$

and the domain D is $s^2 + t^2 \leq 1$. Next,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = (\mathbf{i} - \mathbf{k}) \times (\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

(This was absolutely predictable!).
At last:

$$\text{Area} = \iint_P dA = \iint_D \left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| dA$$
$$= \iint_D \sqrt{6}\, dA = \sqrt{6}\ \text{area of } D$$
$$= \sqrt{6}\,(4\pi) = 4\pi\sqrt{6}.$$

5. Give an iterated integral for the surface integral $\iint_{\mathbf{S}} \mathbf{F} \bullet d\mathbf{S}$, where

$$\mathbf{F} = xy\mathbf{i} + 4x^2\mathbf{j} + yz\mathbf{k},$$

and $\mathbf{S}$ is the surface $z = xe^y$, $0 \leq x,y \leq 1$ with the upward orientation.

$\iint_{\mathbf{S}} \mathbf{F} \bullet d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(\mathbf{s},\mathbf{t})) \bullet \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right) dA$, where $\mathbf{r}$ is a vector description of $\mathbf{S}$, *etc*.
For a vector description, just let $s = x$ and $t = y$. Then,

$$\mathbf{r}(s,t) = s\mathbf{i} + t\mathbf{j} + se^t\mathbf{k}.$$

And so,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = (\mathbf{i} + e^t\mathbf{k}) \times (\mathbf{j} + se^t\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & e^t \\ 0 & 1 & se^t \end{vmatrix} = e^t\mathbf{i} - se^t\mathbf{j} + \mathbf{k}$$

Note the **k** component of this normal is positive, so this is upward pointing, the one we want. Continuing,

$$\iint_S \mathbf{F} \bullet d\mathbf{S} =$$

$$= \int_0^1 \int_0^1 (st\mathbf{i} + 4s^2\mathbf{j} + ste^t\mathbf{k}) \bullet (e^t\mathbf{i} - se^t\mathbf{j} + \mathbf{k}) ds dt$$

$$= \int_0^1 \int_0^1 (ste^t - 4s^3e^t + ste^t) ds dt$$

$$= \int_0^1 \int_0^1 2se^t(t - 2s^2) ds dt$$

6. Suppose $f, g,$ and **F** are nice functions, and that the surface **S** is bounded by the curve **C**. Assume moreover the surface and its boundary are consistently oriented. Show:

a)$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$

Suppose $\mathbf{F} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$. Then,

$$\nabla \times (f\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fp & fq & fr \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}fr - \frac{\partial}{\partial z}fq\right)\mathbf{i} + \left(\frac{\partial}{\partial z}fp - \frac{\partial}{\partial x}fr\right)\mathbf{j} + \left(\frac{\partial}{\partial x}fq - \frac{\partial}{\partial y}fp\right)\mathbf{k}$$

$$= \left(f\frac{\partial r}{\partial y} + r\frac{\partial f}{\partial y} - f\frac{\partial q}{\partial z} - q\frac{\partial f}{\partial z}\right)\mathbf{i} + \left(f\frac{\partial p}{\partial z} + p\frac{\partial f}{\partial z} - f\frac{\partial r}{\partial x} - r\frac{\partial f}{\partial x}\right)\mathbf{j}$$
$$+ \left(f\frac{\partial q}{\partial x} + q\frac{\partial f}{\partial x} - f\frac{\partial p}{\partial y} - p\frac{\partial f}{\partial y}\right)\mathbf{k}$$

$$= f\left[\left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right)\mathbf{j} + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)\mathbf{k}\right]$$
$$+ \left(r\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z}\right)\mathbf{i} + \left(p\frac{\partial f}{\partial z} - r\frac{\partial f}{\partial x}\right)\mathbf{j} + \left(q\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial y}\right)\mathbf{k}$$

$$= f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$$

b)$\nabla \times (\nabla g) = 0$

$$\nabla \times (\nabla g) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 g}{\partial z \partial y} \right)\mathbf{i} + \left(\frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 g}{\partial x \partial z} \right)\mathbf{j} + \left(\frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y \partial x} \right)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

c) $\int_{\mathbf{C}} (f\nabla g) \bullet d\mathbf{r} = \iint_{\mathbf{S}} (\nabla f \times \nabla g) \bullet d\mathbf{S}$

First, use Stokes's Theorem to get

$$\int_{\mathbf{C}} (f\nabla g) \bullet d\mathbf{r} = \iint_{\mathbf{S}} \nabla \times (f\nabla g) \bullet d\mathbf{S}$$

Now, use the result of part a) on the surface integrand, with ∇g playing the role of **F**:

$$\int_{\mathbf{C}} (f\nabla g) \bullet d\mathbf{r} =$$

$$= \iint_{\mathbf{S}} [f\nabla \times (\nabla g) + \nabla f \times \nabla g] \bullet d\mathbf{S}$$

Next, observe that from part b), we know that $\nabla \times (\nabla g) = 0$. Hence,

$$\int_{\mathbf{C}} (f\nabla g) \bullet d\mathbf{r} =$$

$$= \iint_{\mathbf{S}} [\nabla f \times \nabla g] \bullet d\mathbf{S}.$$

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