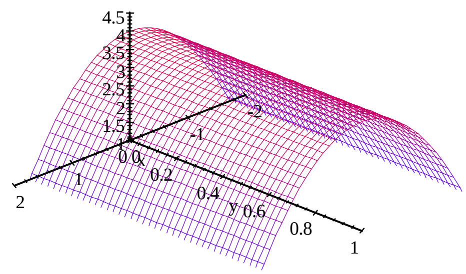
Math 2507B Quiz Two   Solutions

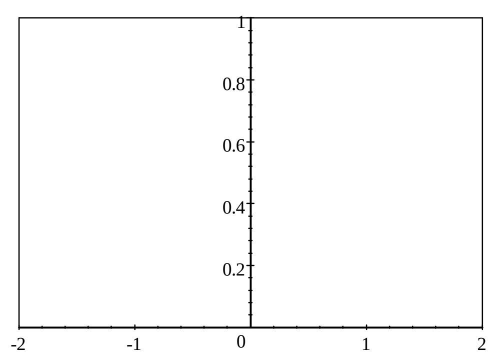
1. Let **V** be the solid bounded above by the surface $z = 4 - x^2$, below by the $x - y$ plane, at one end by the plane $y = 0$, and at the other end by the plane $y = 1$.

a)Give an iterated integral for $\iiint_{\mathbf{V}} z(x + y^2)dV$ in which the first integration is with respect to z.

First, a picture



If we integrate first with respect to z, we have projected the solid onto the x-y plane:

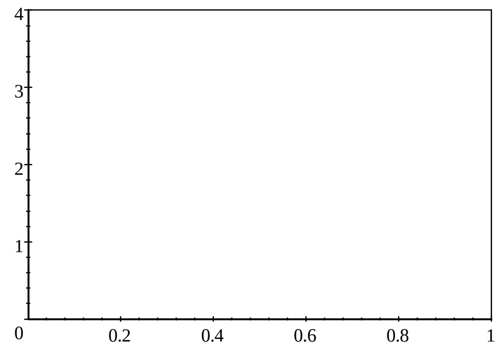


It should be clear from the pictures that

$$\iiint_{\mathbf{V}} z(x + y^2)dV = \int_{-2}^{2}\int_{0}^{1}\int_{0}^{4-x^2} z(x + y^2)dzdydx, \text{ or}$$

$$= \int_{0}^{1}\int_{-2}^{2}\int_{0}^{4-x^2} z(x + y^2)dzdxdy$$

b)Now, suppose the first integration is with respect to x. Here, we project the solid onto the y-z plane:



And we see

$$\iiint_{\mathbf{V}} z(x+y^2)dV = \int_0^1 \int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} z(x+y^2)dxdzdy, \text{ or}$$

$$= \int_0^4 \int_0^1 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} z(x+y^2)dxdydz$$

2. A wire has the shape of the curve $y = x^2$, with one end at (0,0) and the other at (1,1). The density of the wire is $\rho(x,y) = x$. What is the mass of the wire?

We want the scalar line integral $\int_W \rho(x,y)dr$. To find this integral, we need a vector description of the curve. This is easy; simply let $t = x$:

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \text{ for } 0 \le t \le 1.$$

Our mass M is then

$$\int_W \rho(x,y)dr = \int_0^1 \rho(\mathbf{r}(t))|\mathbf{r}'(t)|dt.$$

Now, $\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$, and so $|\mathbf{r}'(t)| = \sqrt{1+4t^2}$. Also, $\rho(\mathbf{r}(t)) = \rho(t,t^2) = t$, and integral becomes

$$Mass = \int_W \rho(x,y)dr = \int_0^1 \rho(\mathbf{r}(t))|\mathbf{r}'(t)|dt = \int_0^1 t\sqrt{1+4t^2}\,dt$$

$$= \left.\frac{(1+4t^2)^{3/2}}{12}\right|_0^1 = \frac{5^{3/2}-1}{12} = \frac{5\sqrt{5}-1}{12}$$

3. Evaluate the integral $\int_H \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (x+1)\mathbf{i} + (y+1)\mathbf{j} + z^2\mathbf{k}$ and H is the helix

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \text{ for } 0 \le t \le 2\pi.$$

This is perfectly straight-forward:

$$\int_H \mathbf{F} \bullet d\mathbf{r} =$$

$$= \int_0^{2\pi} [(\cos t + 1)\mathbf{i} + (\sin t + 1)\mathbf{j} + t^2\mathbf{k}] \bullet (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})dt$$

$$= \int_0^{2\pi} [-\sin t \cos t - \sin t + \sin t \cos t + \cos t + t^2]dt$$

$$= \int_0^{2\pi} [-\sin t + \cos t + t^2]dt = \int_0^{2\pi} t^2 dt = \frac{8\pi^3}{3}$$

4. Find the area of the bounded piece cut from the plane $x + 2y + 3z = 5$ by the cylinder $x^2 + z^2 = 1$.

The area $= \iint_S dA$. To find this integral, we need a vector description of the surface S. This is not hard; simply take $s = x$ and $t = z$:

$$\mathbf{r}(s,t) = s\mathbf{i} + \frac{5 - s - 3t}{2}\mathbf{j} + t\mathbf{k}, \text{ for } s^2 + t^2 \le 1.$$

Now then,

$$\frac{\partial \mathbf{r}}{\partial s} = \mathbf{i} - \frac{1}{2}\mathbf{j}, \text{ and } \frac{\partial \mathbf{r}}{\partial t} = -\frac{3}{2}\mathbf{j} + \mathbf{k}.$$

Hence,

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = -\frac{1}{2}\mathbf{i} - \mathbf{j} - \frac{3}{2}\mathbf{k}, \text{ and so}$$

$$\left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2}.$$

Finally,

$$\text{Area} = \iint_S dA = \iint_D \left| \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} \right| dA$$

$$= \iint_D \frac{\sqrt{14}}{2} dA = \frac{\sqrt{14}}{2} \iint_D dA.$$

But D is the region $s^2 + t^2 \le 1$, and so $\iint_D dA =$area of $D = \pi$. Thus,

$$\text{Area} = \frac{\sqrt{14}}{2} \iint_D dA = \frac{\pi\sqrt{14}}{2}.$$