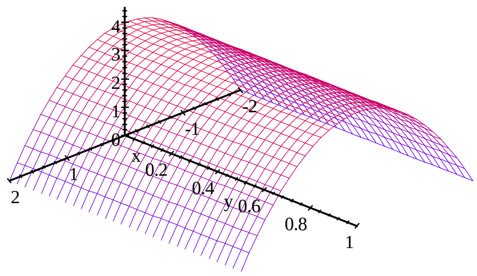
Math 2507B Quiz Three Solutions

1. Find the flux of the function $\mathbf{F}(x,y,z) = x\mathbf{i} + yz^3\mathbf{j} - 3x\mathbf{k}$ upward through the surface cut from the cylinder $z = 4 - x^2$ by the planes $y = 0$, $y = 1$, and $z = 0$.



For a vector description, simply choose $s = x$ and $t = y$:

$$r(s,t) = s\mathbf{i} + t\mathbf{j} + (4 - s^2)\mathbf{k}, \text{ with } -2 \le x \le 2, \text{ and } 0 \le y \le 1.$$

Let's compute

$$\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2s \\ 0 & 1 & 0 \end{vmatrix} = 2s\mathbf{i} + \mathbf{k}.$$

Note that the z component of this vector is always positive, so that this is the orientation specified. Now,

$$\begin{aligned} \iint_S \mathbf{F} \bullet d\mathbf{r} &= \int_{-2}^{2}\int_{0}^{1} \mathbf{F}(\mathbf{r}(s,t)) \bullet \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right) dtds \\ &= \int_{-2}^{2}\int_{0}^{1} [s\mathbf{i} + t(4 - s^2)^3\mathbf{j} - 3s\mathbf{k}] \bullet (2s\mathbf{i} + \mathbf{k}) dtds \\ &= \int_{-2}^{2}\int_{0}^{1} (2s^2 - 3s) dtds = \int_{-2}^{2} (2s^2 - 3s) ds \\ &= \frac{16}{3} + \frac{16}{3} = \frac{32}{3} \end{aligned}$$

2. The solid R is bounded by the surface S, and the outward flux of the function $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 5y\mathbf{j} - z\mathbf{k}$ through the surface S is 102. Find the volume of R.

Gauss's Theorem tells us that

$$\iint_S \mathbf{F} \bullet d\mathbf{r} = \iiint_R div\mathbf{F}dV$$

Thus,

$$\iint_S \mathbf{F} \bullet d\mathbf{r} = 102 = \iiint_R (2+5-1)dV = 6\iiint_R dV.$$

Hence,

$$\text{volume of } R = \iiint_R dV = \frac{102}{6} = 17.$$

3. Let S be the surface consisting of that part of the cylinder $x^2+y^2 = 1$ between the planes $z = 0$ and $z = 1$ together with the hemisphere $x^2+y^2+(z-1)^2 = 1, \quad z \geq 1$. Find the flux outward through S of the function $\mathbf{F}(x,y,z) = ye^z\mathbf{i} + z\mathbf{j} + (z+1)\mathbf{k}$.

If we add a "bottom", $S' = \{(x,y,z) : x^2+y^2 \leq 1\}$ to the surface S, the result is a closed surface $T = S \cup S'$. We now appeal to Gauss's Theorem to get

$$\iiint_R div\mathbf{F}dV = \iint_T \mathbf{F} \bullet d\mathbf{r}.$$

But, $div\mathbf{F} = 1,$ so we have

$$\iint_T \mathbf{F} \bullet d\mathbf{r} = \iint_S \mathbf{F} \bullet d\mathbf{r} + \iint_{S'} \mathbf{F} \bullet d\mathbf{r} = \iiint_R dV.$$

Now, $\iiint_R dV$ is simply the volume of the cylinder together with the volume emclosed by the hemissphere. The cylindrical volume is $\pi r^2 h = \pi$, while the hemispherical volume is $\frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi$. Thus $\iiint_R dV = \pi + \frac{2}{3}\pi = \frac{5}{3}\pi$, and we have

$$\iint_S \mathbf{F} \bullet d\mathbf{r} = \frac{5}{3}\pi - \iint_{S'} \mathbf{F} \bullet d\mathbf{r}$$

Now, $\iint_{S'} \mathbf{F} \bullet d\mathbf{r}$ is easy to compute:

A vector description of S' is simply $\mathbf{r}(s,t) = s\mathbf{i} + t\mathbf{j}$, with domain D the region $s^2+t^2 \leq 1$. The outward pointing normal is just $\mathbf{k}$. Thus

$$\iint_{S'} \mathbf{F} \bullet d\mathbf{r} = \iint_D \mathbf{F}(s,t) \bullet \mathbf{k}dA$$
$$= \iint_D dA = \text{area of } D = \pi.$$

Finally,

$$\iint_S \mathbf{F} \bullet d\mathbf{r} = \frac{5}{3}\pi - \iint_{S'} \mathbf{F} \bullet d\mathbf{r} = \frac{5}{3}\pi - \boldsymbol{\pi} = \frac{2}{3}\pi.$$

4. The quadrilateral Q has vertices (0, 0), (0,5), (5,6), and (4,0), and the triangle T has vertices (1,1), (3,2), and (2,3). Find the are of the region inside Q that is also outside T.

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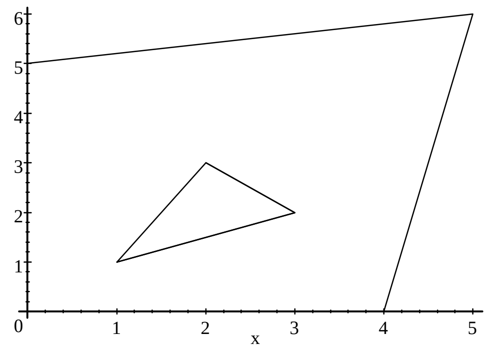

Use the formula we derived from Green's Theorem for the area enclosed by a polygon with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$:

$$area = \frac{1}{2}\left[\sum_{i=1}^{n-1}(x_i + x_{i+1})(y_{i+1} - y_i) + (x_1 + x_n)(y_1 - y_n)\right]$$

Thus the area q of Q is

$$\begin{aligned} q &= \frac{1}{2}[(4+0)(0-0) + (5+4)(6-0) + (5+0)(5-6) + (0+0)(0-5)] \\ &= \frac{49}{2}, \end{aligned}$$

and the area t of the triangle is

$$\begin{aligned} t &= \frac{1}{2}[(1+3)(2-1) + (3+2)(3-2) + (2+1)(1-3)] \\ &= \frac{3}{2}. \end{aligned}$$

Thus the area of the region inside Q and outside T is $\frac{49}{2} - \frac{3}{2} = \frac{46}{2} = 23$.