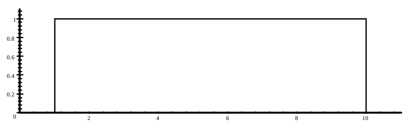
Math 2507B Quiz One Solutions

1. Given that $\int_0^1 \int_1^{10} f(x,y)\,dx\,dy = -4$, $\int_0^2 \int_0^{2x-x^2} f(x,y)\,dy\,dx = 17$,

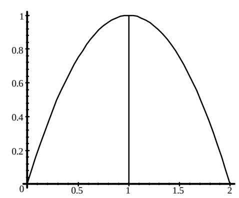
and $\int_0^1 \int_{1-\sqrt{1-y}}^{10} f(x,y)\,dx\,dy = 3$, find $\int_1^2 \int_0^{2x-x^2} f(x,y)\,dy\,dx$.

Let's draw pictures of the regions of integration:

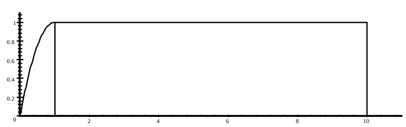
$$\int_0^1 \int_1^{10} f(x,y)\,dx\,dy = \iint_A f(x,y)\,dA :$$



$$\int_0^2 \int_0^{2x-x^2} f(x,y)\,dy\,dx = \iint_B f(x,y)\,dA + \iint_C f(x,y)\,dA :$$



$$\int_0^1 \int_{1-\sqrt{1-y}}^{10} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA :$$



Now, it's clear that $\iint_D f(x,y)\,dA = \iint_B f(x,y)\,dA + \iint_A f(x,y)\,dA$, or

$$3 = \iint_B f(x,y)\,dA + (-4). \text{ Hence}$$

$$\iint_B f(x,y)\,dA = 7.$$

But,

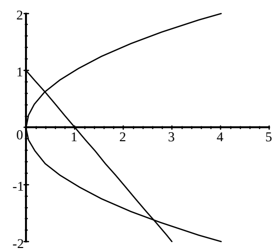
$$\iint_B f(x,y)dA + \iint_C f(x,y)dA = 17, \text{ and so}$$

$$10 + \iint_C f(x,y)dA = 17, \text{ or}$$

$$\iint_C f(x,y)dA = \int_1^2 \int_0^{2x-x^2} f(x,y)dydx = 10$$

2. Give an iterated integral for the area of the region bounded by the curves $x + y = 1$ and $y^2 = x$.

A picture:

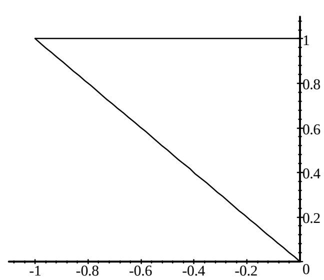


Let's find the points of intersection of these curves. We want $y^2 = x = 1 - y$, or $y^2 + y - 1 = 0$. Solutions are $\left\{y = \frac{1}{2}\sqrt{5} - \frac{1}{2}\right\}$, and $\left\{y = -\frac{1}{2} - \frac{1}{2}\sqrt{5}\right\}$.

Now,

$$\text{Area} = \iint_R dA = \int_{-\frac{1}{2}-\frac{1}{2}\sqrt{5}}^{-\frac{1}{2}+\frac{1}{2}\sqrt{5}} \int_{y^2}^{1-y} dxdy$$

3. Find the integral $\iint_{\mathbf{T}} y^2 e^{xy} dA$, where **T** is the triangle with vertices $(0,0)$, $(0,1)$, and $(-1,1)$.



$$\iint_T y^2 e^{xy} dA = \int_0^1 \int_{-y}^0 y^2 e^{xy} dx dy = \int_0^1 [ye^{xy}]_{-y}^0 dy = \int_0^1 \left(y - ye^{-y^2}\right) dy$$

$$= \frac{y^2}{2} + \frac{e^{-y^2}}{2} \Big|_0^1 = \frac{1}{2} + \frac{e^{-1}}{2} - \frac{1}{2} = \frac{e^{-1}}{2} = \frac{1}{2e}$$

4. Find the centroid of that part of the region $x^2 + y^2 \leq a$ in the first quadrant.

We need

$$\tilde{x} = \frac{\iint_R x dA}{\iint_R dA} \ ; \ \tilde{y} = \frac{\iint_R y dA}{\iint_R dA},$$

where R is the given region. First, we know that $\iint_R dA = \frac{\pi a^2}{4}$, since this is simply one-fourth the are of the circular region. Next,

$$\iint_R x dA = \int_0^a \int_0^{\sqrt{a^2 - x^2}} x dy dx$$

$$= \int_0^a x\sqrt{a^2 - x^2}\, dx = -\frac{(a^2 - x^2)^{3/2}}{3} \Big|_0^a$$

$$= -0 + \frac{a^3}{3} = \frac{a^3}{3}.$$

Thus,

$$\tilde{x} = \frac{\frac{a^3}{3}}{\frac{\pi a^2}{4}} = \frac{4}{3\pi} a.$$

The calculation for $\tilde{y}$ is essentially the same, but just for some variety, let's sue polar coordinates:

$$\iint_R y dA = \int_0^{\pi/2} \int_0^a (r \sin\theta) r dr d\theta = \int_0^{\pi/2} \int_0^a r^2 \sin\theta dr d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} \sin\theta d\theta = \frac{a^3}{3} [-\cos\theta]_0^{\pi/2}$$

$$= \frac{a^3}{3},$$

and so $\tilde{y} = \frac{4}{3\pi}a$ also.