Chapter One– Topological Spaces

Definition. A collection ***T*** of subsets of a set X is a **topology** if

a) ϕ and X are elements of ***T***;

b) the intersection of any two elements of ***T*** is an element of ***T***; and

c) the union of any subcollection of ***T*** is an element of ***T***.

Definition. A set X together with a topology ***T*** is called a **topological space**.

Examples 1.1. Suppose X is a set.

a)The collection of all subsets of X is a topology. This topology is usually called the **discrete** topology.

b)***T***= $\{\phi, X\}$ is a topology. This one is usually called the **trivial** topology.

c)***T*** $=\{U \subset X : X \setminus U$ is finite$\} \cup \{\phi\}$ is a topology, called the **cofinite** topology.

d)***T*** $=\{U \subset X : X \setminus U$ is countable$\} \cup \{\phi\}$ is a topology. (You guess its name.)

e)Suppose $A \subset X$,. Then ***T***= $\{\{\phi, A, X\}$ is a topology. To the best of my knowledge, this one does not have a name.

Definition. If $(X,$ ***T***$)$ is a topological space, and A is a subset of X, a point $p \in X$ is an **accumulation point** of A if every $U \in$ ***T*** with $p \in U$ meets A in a point other than p.

Definitions. The set A' of all accumulation points of A is called the **derived set** of A. The set $A \cup A'$ is called the **closure** of A and is traditionally denoted clA.

Definition. If $A = clA$, then A is called a **closed** set.

Proposition 1.2. A set A is closed if and only if $A' \subset A$.

Definition. If $X \setminus A$ is closed, then A is called an **open** set.

Theorem 1.3. If $(X,$ ***T***$)$ is a topological space, a set $U \subset X$ is open if and only if $U \in$ ***T***.

Theorem 1.4. Suppose $(X,$ ***T***$)$ is a topological space. Then:

a)ϕ and X are closed sets;

b)the union of two closed sets is a closed set; and

c)the intersection of any collection of closed sets is a closed set.

Definition. A set N is a **neighborhood of the point** p if there is an open set U such that $p \in U \subset N$. A set N is a **neighborhood of the set** A if it is a neighborhood of each point of A.

Theorem 1.5. A point p is a member of the closure of a set A if and only if every neighborhood of p meets A.

Proposition 1.6. If $A \subset B$, then $clA \subset clB$.

Proposition 1.7. If F is a closed set and $A \subset F$, then $clA \subset F$.

Theorem 1.8. The closure of a set is a closed set.

Theorem 1.9. $clA = \bigcap \{F: A \subset F, \text{ and } F \text{ is closed}\}$.

Definition. The **interior** of a set A is the set of all p such that A is a neighborhood of p. It is usually denoted $intA$.

Proposition 1.10. If U is an open set and $U \subset A$, then $U \subset intA$.

Theorem 1.11. The interior of a set is an open set.

Theorem 1.12. $intA = \bigcup \{U : \ U \subset A, \text{ and } U \text{ is open}\}$.

Definition. If A is a subset of the space $(X,\boldsymbol{T})$, then the **boundary** of A is the set $FrA = (clA) \cap cl(X \setminus A)$.

Definition. A subset A of the space $(X,\boldsymbol{T})$ is said to be **dense** if $clA = X$.

Definition. Let $(X, \boldsymbol{T})$ be a topological space. A subset $\boldsymbol{B} \subset \boldsymbol{T}$ such that every element of $\boldsymbol{T}$ is a union of elements of $\boldsymbol{B}$ is a **base** for $\boldsymbol{T}$.

Example 1.13. For any set X, the collection $\boldsymbol{B} = \{\{x\} : x \in X\}$ is a base for the discrete topology.

Proposition 1.14. Suppose $\boldsymbol{B}$ is a base for $\boldsymbol{T}$. Then a point p is an accumulation point of a set A if and only if every member of $\boldsymbol{B}$ containing p meets A in a point other than p.

Corollary 1.15. A point p is a member of clA if and only if every element of $\boldsymbol{B}$ containing p meets A.

Theorem 1.16. Suppose X is a set, and $\boldsymbol{B}$ is a collection of subsets of X such that

a) $X = \cup\boldsymbol{B}$; and

b) if B_1 and B_2 are elements of $\boldsymbol{B}$ and $p \in B_1 \cap B_2$, then there is a $B_3 \in \boldsymbol{B}$ so that $p \in B_3 \subset B_1 \cap B_2$.

Then the collection of all unions of elements of $\boldsymbol{B}$ is a topology for X. That is, $\boldsymbol{B}$ is a base for a topology for X.

Proposition 1.17. Suppose $\boldsymbol{T}^*$ is a collection of topologies for X. Then $\cap\boldsymbol{T}^*$ is a topology for X.

Definition. Let X be a set and let $\boldsymbol{C}$ be a collection of subsets of X. The topology

$$\boldsymbol{T}_{\boldsymbol{C}} = \cap\,\{\boldsymbol{T} : \boldsymbol{C} \subset \boldsymbol{T} \text{ and } \boldsymbol{T} \text{ is a topology for } X\}$$

is the topology **generated by** $\boldsymbol{C}$.

Proposition 1.18. If $\boldsymbol{C}$ is a collection of subsets of the set X, then

$$\boldsymbol{B} = \{\cap\boldsymbol{F} : \boldsymbol{F} \subset \boldsymbol{C} \text{ and } \boldsymbol{F} \text{ is finite}\}$$

is a base for the topology generated by $\boldsymbol{C}$.

Examples 1.19.

a)Let X be the real numbers and let $\boldsymbol{B} = \{(a,b) : a < b\}$. Then $\boldsymbol{B}$ is a base for a topology for X. [Here, $(a,b) = \{x : a < x < b\}$.

b)Let X be the reals and let

$$\boldsymbol{C} = \{(-\infty,a) : a \in X\} \cup \{(a,\infty) : a \in X\}.$$

The topology generated by $\boldsymbol{C}$ is the same as the topology in a).

Definition. Suppose X is a set. A real-valued function $\rho : X \times X \to \boldsymbol{R}$ is called a **pseudometric** if it is true that for all x, y, and z :

a)$\rho(x,y) \geq 0$;
b)$\rho(x,y) = \rho(y,x)$;
c)$\rho(x,x) = 0$; and
d)$\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.

The pair (X,ρ) is called a **pseudometric space.**

Definition. Suppose (X,ρ) is a pseudometric space, and suppose $x \in X$ and r is a real number. The set

$$C(x;r) = \{y \in X : \rho(x,t) < r\}$$

is a **cell.** The point x is the **center** of the cell, and r is the **radius**.

Examples 1.20.

a)For any set X, the function $\rho : X \times X \to \boldsymbol{R}$ defined by $\rho(x,y) = 0$ is a pseudometric.
b) For any set X, the function

$$\rho(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$$

is a pseudometric.

c)If X is the real numbers, then $\rho(x,y) = |x - y|$ is a pseudometric.

Theorem 1.21. Let (X, ρ) be a pseudometric space. Then the collection $\boldsymbol{B}$ of all cells is a base for a topology.

Definition. The topology in the preceding theorem is called the topology **generated** by the pseudometric ρ. Two pseudometrics for a set X are **equivalent** if they generate the same topology.

Examples 1.22.

a)The pseudometric in Example 1.21a generates the trivial topology (Example 1.1b).

b)The pseudometric in Example 1.21b generates the discrete topology (Example 1.1a).

c)The topology generated by the pseudometric in Example 1.21c is called the **usual** topology for the reals.

Proposition 1.23. Suppose $(X, \boldsymbol{T})$ is a topological space and $A \subset X$. Then the collection $\boldsymbol{T}_A = \{U \cap A : U \in \boldsymbol{T}\}$ is a topology for A. [The space $(A, \boldsymbol{T}_A)$ is called a **subspace** of $(X, \boldsymbol{T})$ and $\boldsymbol{T}_A$ is called the **subspace**, or **relative**, topology.]

Proposition 1.24. Suppose $(A, \boldsymbol{T}_A)$ is a subspace of $(X, \boldsymbol{T})$. Then $S \subset A$ is closed with respect to the subspace topology if and only if there is a closed set $F \subset X$ such that $S = F \cap A$.

Proposition 1.25. Suppose $(A, \boldsymbol{T}_A)$ is a subspace of $(X, \boldsymbol{T})$, and suppose $\boldsymbol{B}$ is a base for $\boldsymbol{T}$. Then $\boldsymbol{B}_A = \{B \cap A : B \in \boldsymbol{B}\}$ is a base for $\boldsymbol{T}_A$.

Proposition 1.26. Suppose $(A, \boldsymbol{T}_A)$ is a subspace of $(X, \boldsymbol{T})$, and suppose $\boldsymbol{T}$ is generated by the pseudometric ρ. Then $\boldsymbol{T}_A$ is generated by the restriction $\rho_A = \rho|A \times A$.

Examples 1.27.

Let Q be the rationals with the topology it inherits as a subspace of the reals with the usual topology.

a)The set $S = \{x \in Q : -1 < x < 1\}$ is open in the subspace topology but not closed, while

b)the set $V = \{x \in Q : -\sqrt{2} < x < \sqrt{2}\}$ is both open and closed in this topology.