Chapter Ten - Convex Sets, Simplices, and All That

Definition. Suppose S is a subset of a real linear space. A **weighted mean** of S is a linear combination $\sum_{i=0}^{k} t_i s_i$, in which $s_i \in S$, and $t_i \geq 0$ for $i = 0, 1, \ldots, k$ and $\sum_{i=0}^{k} t_i = 1$.

Proposition 10.1. The collection of all weighted means of a set is a convex set.

Proposition 10.2. If S is convex and x is a weighted mean of S, then $x \in S$.

Definition. Suppose S is a subset of a real linear space X. The set

$$[S] = \cap\{K \subset X : K \supset S, \text{ and } K \text{ convex}\}$$

is called the **convex hull** of S.

Theorem 10.3. Suppose S is a subset of a real linear space. The convex hull of S is the set of all weighted means of S.

Theorem 10.4. Suppose K is a convex subset of a topological linear space. Then the closure of K is also convex.

Definition. The **convex closure** of a subset of a topological linear space is the closure of its convex hull.

Theorem 10.5. The convex closure of S is the intersection of the collection of all closed convex supersets of S.

Theorem 10.6. Suppose S is a compact subset of $\mathbf{R}^n$ with the usual topology. Then the convex hull of S is compact.

Definition. The convex hull of a finite subset of $\mathbf{R}^n$ is a **convex polyhedron.**

Definition. Suppose $\boldsymbol{F} = \{a_0, a_1, \ldots, a_k\}$ is a linearly independent subset of $\mathbf{R}^n$. The convex hull of $\boldsymbol{F}$ is a **k-simplex** and the a_i are known as the **vertices** of the k-simplex.

Definition. If $[\{a_0, a_1, \ldots, a_k\}]$ is a k-simplex, the convex hull of a subset of $\{a_0, a_1, \ldots, a_k\}$ is a **face** of the simplex.

Definition. If $S_k = [\boldsymbol{F}]$ is a k-simplex and $\boldsymbol{G} \subset \boldsymbol{F}$ is a proper subset of $\boldsymbol{F}$, then $[\boldsymbol{G}]$ is a **proper face** of S_k and $[\boldsymbol{F} \backslash \boldsymbol{G}]$ is the **face opposite** $\boldsymbol{G}$.

Theorem 10.7. Suppose $S_k = [\{a_0, a_1, \dots, a_k\}]$ is a k-simplex. Then

$$S_k = \left\{ \sum_{i=0}^{k} t_i a_i \; : \; t_i \geq 0, \text{ and } \sum_{i=0}^{k} t_i = 1 \right\}.$$

Moreover, if for $x \in S_k$,

$$x = \sum_{i=0}^{k} t_i a_i = \sum_{i=0}^{k} s_i a_i,$$

then $t_i = s_i$ for all $i = 0, 1, \dots, k$.

Definition. Suppose $S_k = [\{a_0, a_1, \dots, a_k\}]$ is a k-simplex and $x \in S_k$. Then the unique t_i such that $x = \sum_{i=0}^{k} t_i a_i$ are the **barycentric coordinates** of x.

Definition. If $S_k = [\{a_0, a_1, \dots, a_k\}]$ is a k-simplex, the set

$$T_k = \left\{ \sum_{i=0}^{k} t_i a_i \; : \; t_i > 0, \text{ and } \sum_{i=0}^{k} t_i = 1 \right\}$$

is called an **open k-simplex**, and is denoted $(\{a_0, a_1, \dots, a_k\})$.

Note. It is customary to omit the curly brackets and write $[a_0, a_1, \dots, a_k]$ instead of $[\{a_0, a_1, \dots, a_k\}]$ and $(a_0, a_1, \dots, a_k)$ instead of $(\{a_0, a_1, \dots, a_k\})$. We shall follow this very sensible custom.

Definition. If $T_k = (\boldsymbol{F})$ is an open simplex and $\boldsymbol{G} \subset \boldsymbol{F}$, then $(\boldsymbol{G})$ is called a **face** of T_k.

Definition. Suppose S_k is a simplex. An **open face** of S_k is a face of the open simplex having the same vertices as S_k.

Proposition 10.8. Suppose S_k is a simplex and $x \in S_k$. Then x is contained in exactly one open face of S_k.

Definition. Suppose S_k is a simplex. A **triangulation** of S_k is a collection $\boldsymbol{T}$ of open simplices such that

i) $S_k = \cup \boldsymbol{T}$;

ii) if $T \in \boldsymbol{T}$ and R is a face of T, then $R \in \boldsymbol{T}$; and

iii) the collection $\boldsymbol{T}$ is pairwise disjoint.

Theorem 10.9. Let $\boldsymbol{T}$ be a triangulation of the simplex $S = [a_0, a_1, \dots, a_k]$, and let $\boldsymbol{V}$ be the set of all vertices of $\boldsymbol{T}$. Suppose $f : \boldsymbol{V} \to \{a_0, a_1, \dots, a_k\}$ is a function having the property that for each $v \in \boldsymbol{V}$, $f(v)$ is a vertex of the open face of S which contains v. Then there is an element $T = (F) \in \boldsymbol{T}$ such that $f(F) = \{a_0, a_1, \dots, a_k\}$.

Note. This is the celebrated **Sperner's Lemma**.

Proposition 10.10. Suppose $\boldsymbol{F}$ is a finite collection of closed subsets of a compact pseudometric space X such that $X = \cup\boldsymbol{F}$. Then there is a $\delta > 0$ so that if $A \subset X$ has diameter $< \delta$, then $\cap\{F \in \boldsymbol{F} : F \cap A \neq \emptyset\} \neq \emptyset$.

Proposition 10.11. If S is a simplex and $\varepsilon > 0$, then there is a triangulation $\boldsymbol{T}$ of S every element of which has diameter $< \varepsilon$.

Proposition 10.12. Let $S = [a_0, a_1, \ldots, a_k]$ be a simplex, and let $F_0, F_1, \ldots, F_k$ be closed subsets of S such that for each $\{i_1, i_2, \ldots, i_l\} \subset \{0, 1, \ldots, k\}$, it is true that $[a_{i_1}, a_{i_2}, \ldots, a_{i_l}] \subset F_{i_1} \cup F_{i_2} \cup \ldots \cup F_{i_l}$. Then $\cap\{F_i : i = 0, 1, \ldots, k\} \neq \emptyset$.

Note. This is yet another famous result: the **Knaster-Kuratowski-Mazurkiewicz Covering Theorem.**

Proposition 10.13. Let $S = [a_0, a_1, \ldots, a_k]$. For each $j = 0, 1, \ldots, k$, define the function $c_j : S \to [0,1]$ by $c_j(x) = c_j\left(\sum_{i=0}^{k} t_i a_i\right) = t_j$. Then each c_j is continuous.

Proposition 10.14. Suppose S is a k-simplex and $f : S \to S$ is a continuous function. For each $i = 0, 1, \ldots, k$, let $F_i = \{x \in S : c_i(f(x)) \leq c_i(x)\}$, where c_i is defined in the previous proposition. Then $\cap\{F_i : i = 0, 1, \ldots, k\} \neq \emptyset$.

Theorem 10.15. Suppose S is a k-simplex and $f : S \to S$ is continuous. Then $f(x) = x$ for some $x \in S$.

Note. This is the **Brouwer Fixed Point Theorem**, one of the most celebrated of all theorems.