Chapter Eleven - Quotient Spaces

Proposition 11.1. Let $f : X \to Y$ be a function from a topological space $(X, \boldsymbol{T})$ into a set Y. Then the collection $\boldsymbol{S} = \{U \subset Y : f^{-1}(U) \in \boldsymbol{T}\}$ is a topology for Y.

Definition. The topology $\boldsymbol{S}$ in the previous proposition is the **strong topology by** f. In case $Y = f(X)$, the strong topology by f is called the **quotient topology by** f.

Proposition 11.2. Suppose $f : X \to Y$ is a function from a topological space $(X, \boldsymbol{T})$ into a topological space $(Y, \boldsymbol{V})$, and let $\boldsymbol{S}$ be the strong topology by f. Then f is continuous if and only if $\boldsymbol{V} \subset \boldsymbol{S}$.

Definitions. A function $f : X \to f(X) = Y$ from one topological space onto another is **open** if $f(U)$ is open whenever U is open. It is **closed** if $f(F)$ is closed whenever F is closed.

Proposition 11.3. Suppose $(X, \boldsymbol{T})$ and $(Y, \boldsymbol{V})$ are topological spaces and suppose $f : X \to f(X) = Y$ is an open function. If $\boldsymbol{S}$ is the quotient(=strong) topology by f, then $\boldsymbol{S} \subset \boldsymbol{V}$.

Proposition 11.4. Suppose $(X, \boldsymbol{T})$ and $(Y, \boldsymbol{V})$ are topological spaces and suppose $f : X \to f(X) = Y$ is a closed function. If $\boldsymbol{S}$ is the quotient topology by f, then $\boldsymbol{S} \subset \boldsymbol{V}$.

Definition. A continuous function $f : X \to f(X) = Y$ from one topological space onto another is called a **quotient map** if the topology of Y is the quotient topology by f.

Theorem 11.5. Every open or closed continuous function from one topological space onto another is a quotient map.

Proposition 11.6. Suppose $f : X \to f(X) = Y$ is continuous, where X is a compact topological space, and Y is a Hausdorff space. Then f is a quotient map.

Definitions. Suppose $\boldsymbol{D}$ is a collection of subsets of a set X. If the elements of $\boldsymbol{D}$ are pairwise disjoint, and if $X = \cup\boldsymbol{D}$, then $\boldsymbol{D}$ is called a **decomposition** of X. The function $p : X \to \boldsymbol{D}$ define by $p(x) = D$, where D is the unique element of $\boldsymbol{D}$ such that $x \in D$ is called the **natural map** associated with $\boldsymbol{D}$.

If R is an equivalence relation on a set X, the collection of all the equivalence classes is a decomposition called the **decomposition induced by** R. In this case the decomposition is denoted X/R.

Definition. Suppose X is a topological space, and suppose R is an equivalence relation on X. The set X/R endowed with the quotient topology by the natural map is called the **quotient space** determined by X and R. A quotient space is also frequently called a **decomposition space**, or sometimes an **identification space**

Definition. Let $f : X \to f(X) = Y$ be a function from one topological space onto another and let

$$R = \{(x,y) \in X \times X : f(x) = f(y)\}.$$

It is clear that R is an equivalence relation on X. The quotient space X/R is called the **point inverse decomposition by** f and is denoted X/f.

Theorem 11.7. Suppose $f : X \to f(X) = Y$ is continuous, and let $p : X \to X/f$ be the natural map onto the point inverse decomposition by f. The the function

$$h : X/f \to Y$$

given by $h(p(x)) = f(x)$ is a homeomorphism if and only if f is a quotient map.