Chapter Twelve - Surfaces

Definition. A Hausdorff space in which each point has an open neighborhood homeomorphic with $\mathbf{R}^n$ is an **n-manifold.**

Definition. A connected 2-manifold is called a **surface**.

Examples. Convince yourself that each of the following topological spaces is a surface.
a)$\mathbf{R}^2$
b)$S = \{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$.

Another Example. Let X be the square $[0,1] \times [0,1]$ with the topology it inherits from the plane with the usual topology. Define a decomposition, or equivalence relation R, on X by listing the equivalence classes $p((x,y))$, where p is the natural map of X onto X/R:

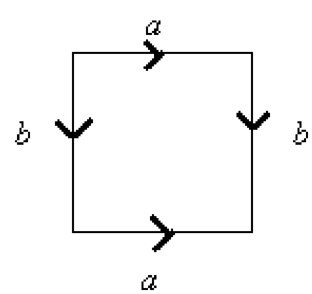
$$p((x,y)) = \{(x,y)\} \text{ for } 0 < x < 1 \text{ and } 0 < y < 1;$$

$$p((x,0)) = p((x,1)) = \{(x,0),(x,1)\} \text{ for } 0 \leq x \leq 1;$$

$$p((0,y)) = p((1,y)) = \{(0,y),(1,y)\} \text{ for } 0 \leq y \leq 1.$$

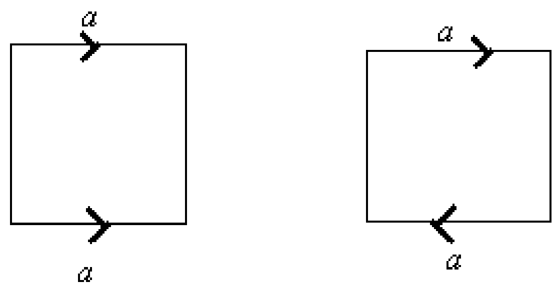
Then the quotient space X/R is a surface.

Note. The decomposition described in the previous example can be indicated by the diagram:



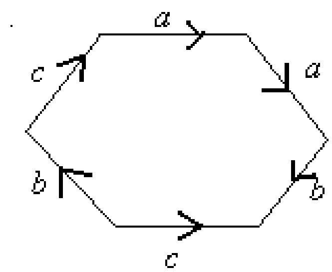
Sides that are to be identified are labeled with the same letter of the alphabet, and the identifications should be made so that the directions indicated by the arrows agree.

Exercises. What are the quotient spaces described by the following diagrams?



Proposition 12.1. Let X be a polygon with an even number of sides (with the topology inherited from the plane with the usual topology). The quotient space obtained by identifying the sides in pairs is a surface.

Example.



Definition. Let S be a compact surface. A **triangulation** of S is a finite collection $\boldsymbol{T} = \{T_1, T_2, \ldots, T_n\}$ of closed subsets of S such that
i)$S = \cup \boldsymbol{T}$;
ii)each $T \in \boldsymbol{T}$ is homeomorphic with a 2-simplex; and
iii)If T_i and T_j are in $\boldsymbol{T}$, then $T_i \cap T_j$ is either a)empty, b)a vertex of each, or an edge of each. [A vertex of T is the image of a vertex of the 2-simplex under the homeomorphism in ii), and an edge is the image of an edge of the 2-simplex, *etc.*]

The elements of $\boldsymbol{T}$ are called triangles.

Theorem 12.2. Every compact surface has a triangulation.

Proposition 12.3. Let $\boldsymbol{T}$ be a triangulation of the compact surface S. The triangles $T \in \boldsymbol{T}$ can be numbered so that if $\boldsymbol{T} = \{T_1, T_2, \ldots, T_n\}$, then the triangle T_i has an edge in common with at least one of the triangles $T_1, T_2, \ldots, T_{i-1}$, for $2 \leq i \leq n$.

Proposition 12.4. Let $\boldsymbol{T}$ be a triangulation of the compact surface S. For each $T_i \in \boldsymbol{T}$, let $\varphi_i : T_i' \to T_i$ be a homeomorphism from a 2-simplex (i.e., a triangular shaped closed set) in the plane onto T_i. Let $T' = \cup\{T_i' : i = 1, 2, \ldots, n\}$. Then the function $\varphi : T' \to S$ defined by $\varphi|T_i' = \varphi_i$ is a quotient map.

Proposition 12.5. For each $i = 2, 3, \ldots, n$, let e_i be an edge of T_i and one other triangle T_j, where $1 \leq j < i$. In the space T', let R be the relation

$$R = \{(x,y) \in \cup\{e_i : i = 2,3,\ldots,n\} \subset T' : \varphi(x) = \varphi(y)\}.$$

Then the quotient space T'/R is a closed disc in the plane.

Theorem 12.6. Every compact surface is the quotient obtained by identifying the sides of a polygon with an even number of sides in pairs.

Example. A triangulation can be describe by labeling the vertices and then specifying the triangles in the triangulation by simply listing triples of these labels–a triple of vertices being the label for a triangle. Specifically, suppose we have the following triangulation of a compact surface:

123 234 341 412

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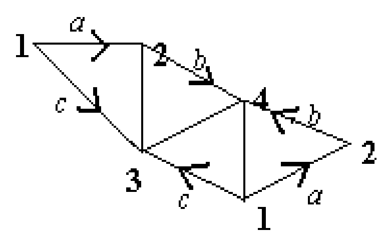
(Here we have used positive integers to label the vertices.) Let's construct the polygon T'/R.



Can you indentify this surface?