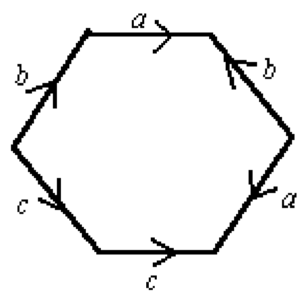
Chapter Thirteen - More Surfaces

Notation. In the previous chapter we saw that each compact surface is a quotient of a polygon with the edges identified in pairs. An easy and convenient method of specifying which edges are to be identified is to begin at any vertex and proceed around the boundary of the polygon, recording the letters assigned to the different sides in succession. If the arrow on a side points in the same direction we are going around the boundary, then we simply write the letter for that side with no exponent. If the arrow points in the opposite direction, we write the letter with the exponent -1.

Example. The quotient indicated in the picture is $ab^{-1}ac^{-1}c^{-1}b$.



Note there are many other symbols for the same surface. If we started at some other vertex, we would obtain a cyclic permutation of the one given, and the difference is, I hope, oblivious were one to travel around the boundary in the opposite direction.

Definition. Let S_1 and S_2 be disjoint surfaces. Choose two closed discs $D_1 \subset S_1$ and $D_2 \subset S_2$. Let $S_i' = S \backslash intD_i$ for $i = 1,2$. Now choose a homeomorphism h from the boundary circle of D_1 onto the boundary circle of D_2. The **connected sum** of S_1 and S_2 is the quotient space of $S_1' \cup S_2'$ obtained by identifying the points x and $h(x)$ for all x in the boundary of D_1. The connected sum of S_1 and S_2 is usually denoted $S_1\#S_2$.

In more down to Earth language, the connected sum of S_1 and S_2 is the surface obtained by cutting a small circular hole in each surface and then gluing the two surfaces together along the boundaries of the circles.

Remark. You need to convince yourself that $S_1\#S_2$ is a surface which depends not on the choice of the discs D_1 and D_2 or the choice of the homeomorphism h.

Proposition 13.1. If the compact surface S_1 is described by the symbol $x_1x_2\ldots x_k$ and the compact surface S_2 is described by $y_1y_2\ldots y_n$, then the connected sum $S_1\#S_2$ is described by $x_1x_2\ldots x_ky_1y_2\ldots y_n$.

Definition. Let S be the surface resulting from identifying the sides of the polygon D in pairs. In a symbol describing these identifications, the symbol designating a certain pair of edges occurs with both exponents $+1$ and -1, we call that pair of edges a pair of the **first kind**. Otherwise the pair is a pair of the **second kind**.

Proposition 13.2. Suppose S is described by $aa^{-1}x_1x_2\ldots x_k$. Then S is described by $x_1x_2\ldots x_k$. (We sometimes say simply that two adjacent edges of the first kind may be eliminated.)

Proposition 13.3. There is a polygon description of the compact surface S in which all vertices are identified to just one vertex.

Proposition 13.4. There is a polygon description of the compact surface S in which any pair of edges of the second kind is adjacent.

Proposition 13.5. Let P be a projective plane and let T be a torus. Then

$$P\#T = P\#P\#P.$$

Proposition 13.6. Suppose the compact surface S is described by the symbol $x_1x_2\ldots x_k$ in which all pairs of edges of the second kind are adjacent. If there is a pair a and $a^{-1}$of the first kind, then there must be another pair of the first kind, b and b^{-1} so that $x_1x_2\ldots a\ldots b\ldots a^{-1}\ldots b^{-1}\ldots x_m$.

Proposition 13.7. In the previous proposition, there is a description of S in which the pairs of pairs of of the first kind occur as $aba^{-1}b^{-1}$.

Theorem 13.8. Any compact surface is either
i)a sphere;
ii)a connected sum of tori; or
iii)a connected sum of projective planes.

Definition. The **genus** of a compact surface is the number of projective planes or the number of tori of which it is a connected sum. The genus of a sphere is 0.

Definition. Let $\boldsymbol{T}$ be a triangulation of the compact surface S. Let v be the number of vertices, e be the number of edges, and t the number of triangles. Then

$$\chi(S) = v - e + t$$

is the **Euler characteristic** of S.

Remark. It is, of course, not at all obvious that it make sense to speak of the Euler characteristic of a surface S when the definition depends of a triangulation of S. A most remarkable result is the fact that this number is indeed independent of the triangulation.

Proposition 13.9. The Euler characteristic is indeed independent of the specific triangulation used in its computation.

Proposition 13.10. The Euler characteristic of a sphere is 2.

Theorem 13.11. Let S_1 and S_2 be compact surfaces. Then

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proposition 13.12. If S is a connected sum of n tori, then $\chi(S) = 2 - 2n$.

Proposition 13.13. If S is a connected sum of n projective planes, then $\chi(S) = 2 - n$.

Theorem 13.14. Suppose S is a compact n-manifold. Then there is an m so that S is a subspace of the Euclidean space $\mathbf{R}^m$.