Chapter Fourteen - Pseudormed Linear Spaces

Definition. Suppose X is a linear space. A **pseudonorm** for X is a real-valued function $p : X \to \mathbf{R}$ such that
i) $p(x) \geq 0$ for all $x \in X$;
ii) $p(0) = 0$;
iii) $p(tx) = |t|p(x)$ for all scalars t and $x \in X$;
iv) $p(x+y) \leq p(x) + p(y)$.

Proposition 14.1. If X is a linear space and p is a pseudonorm for X, then the function $d : X \times X \to \mathbf{R}$ defined by $d(x,y) = p(x-y)$ is a pseudometric.[It is usually called the pseudometric generated by p. The topology generated by by the pseudometric d is also said to be generated by the pseudonorm p.]

Proposition 14.2. The topology on a linear space X generated by a pseudonorm is a linear space topology for X; that is, X endowed with this topology is a topological linear space. [In this case, X is called a **pseudonormed linear space**.].

Definition. A pseudonorm p such that $p(x) = 0$ only if $x = 0$ is called a **norm**. A linear space endowed with a norm is a **normed linear space.**

Note. Where there is no danger of confusion, $p(x)$ is usually denoted $|x|$.

Proposition 14.3. A pseudometric generated by a norm is a metric.

Definition. A subset S of a pseudonormed space is said to be **bounded** if there is an M so that $|s| \leq M$ for every $s \in S$.

Theorem 14.4. Suppose $f : X \to Y$ is a linear function from from one pseudonormed linear space into another. Let $B = \{x \in X : |x| \leq 1\}$. Then f is continuous if and only if $f(B)$ is bounded. [The topologies involved are the ones generated by the pseudonorms.]

Proposition 14.5. If X is a linear space, the collection of all linear functionals on X with the obvious definitions of addition and scalar multiplication is a linear space $X^{\#}$.

Definition. If X is a topological linear space, the subspace X' of $X^{\#}$ consisting of the continuous linear functionals is known as the **dual space** of X.

Theorem 14.6. If X a pseudonormed space, the function $q : X' \to \mathbf{R}$ defined by

$$q(f) = \sup\{|f(x)| : |x| \leq 1\}$$

is a norm for X'. [The space X' endowed with this norm is called the **conjugate space** of X, and is usually denoted X^*. It is also traditional to write $q(f)$ as $|f|$.]

Note. The vertical bar $|\cdot|$ has many different meanings. The context will generally make it clear which is intended.

Proposition 14.7. If f a continuous linear functional on the pseudonormed space X, then $|f(x)| \leq |f||x|$ for all $x \in X$.

Theorem 14.8. If X is a pseudonormed space, the conjugate space X^* is a complete space.

Definition. A complete normed linear space is a **Banach space.**

Proposition 14.9. Let $a \in X$, a pseudonormed linear space. Define the function $\widehat{a} : X^* \to \mathbf{R}$ by $\widehat{a}(f) = f(a)$. Then $\widehat{a}$ is a continuous linear functional on X^*.(Or, in other words, $\widehat{a} \in X^{**}$.)

Proposition 14.10. Suppose X is a normed real linear space and suppose $a, b \in X$ and $a \neq b$. The there is a continuous linear functional f on X such that $f(a) \neq f(b)$.

Proposition 14.11. Suppose X is a normed real linear space. If $a \in X$ and $a \neq 0$, then there is a continuous linear functional f on X so that $f(a) = |a|$ and $|f| = 1$.

Theorem 14.12. Let X be a pseudonormed real linear space and let $h : X \to X^{**}$ be defined by $h(a) = \widehat{a}$. Then h is a linear homeomorphism into X^{**}.[The linear homeomorphism h is called the **natural embedding** of X into X^{**}.]

Definition. A pseudonormed linear space X is said to be **reflexive** if $h(X) = X^{**}$.

Proposition 14.13. A reflexive pseudonormed linear space is a Banach space.

Proposition 14.14. If X is a finite dimensional real linear space, then $\dim X = \dim X^{\#}$.

Theorem 14.15. If X is a finite dimensional normed real linear space, then

$$\dim X = \dim X^* = \dim X^{**}.$$

Theorem 14.16. A finite dimensional normed real linear space is reflexive, and hence complete.

Theorem 14.17. Every linear functional on a finite dimensional real linear space is continuous.

Proposition 14.18. Let $a = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$, and let p be defined by

$$p(a) = \sum_{i=1}^{n} |a_i|.$$

Then p is a norm for $\mathbf{R}^n$.

Proposition 14.19. Suppose $\{e_1, e_2, \ldots, e_n\}$ is a base for the normed linear space X, and let $h : \mathbf{R}^n \to X$ be defined by

$$h(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} a_i e_i.$$

Then h is a linear homeomorphism of $\mathbf{R}^n$ onto X.

Theorem 14.20. Any two n –dimensional normed linear spaces are linearly homeomorphic.

Corollary 14.21. All norms for a finite dimensional linear space generate the same topology.