Chapter Two –Continuous Functions

Definition. A function $f : (X, \boldsymbol{T}) \to (Y, \boldsymbol{S})$ from one topological space into another is **continuous** if $f(clA) \subset clf(A)$ for every $A \subset X$.

Comment. Note that the continuity of a function depends on the topologies assigned to the domain and range, so it is an abuse of language to speak of a function's being continuous without reference to the topologies under consideration. It is, however, in most cases obvious which topologies are involved. When this is not the case, explicit reference to the topologies of the domain and range must be made.

Theorem 2.1. A function from one topological space to another is continuous if and only if the inverse image of every closed set is closed.

Theorem 2.2. A function from one topological space to another is continuous if and only if the inverse image of every open set is open.

Theorem 2.3. Suppose $f : (X, \rho) \to (Y, \sigma)$ is a function from one pseudometric space into another. Then these are equivalent:
a)f is continuous.
b)For each $x \in X$ and $\varepsilon > 0$, there is a δ so that $\sigma(f(x), f(y)) < \varepsilon$ whenever $\rho(x, y) < \delta$.

Proposition 2.4. A function $f : X \to Y$ from one topological space into another is continuous if and only if $f : X \to f(Y)$ is continuous.

Proposition 2.5. Suppose $f : X \to Y$ from one topological space into another is continuous, and suppose $A \subset X$. Then the restriction $f|A$ is continuous.

Examples 2.6.
a)Suppose the set Y is endowed with the trivial topology. Then any function from a topological space into Y is continuous.
b)Suppose the set X is endowed with the discrete topology. Then any function from X into a topological space is continuous.

Definition. A one-to-one function $f : X \to f(X) = Y$ from one topological space onto another is a **homeomorphism** if both it and its inverse are continuous.

Definition. Two topological spaces are **homeomorphic** if there is a homeomorphism from one onto the other.

Definition. Suppose X is a set and $\{(Y_a, \boldsymbol{T}_a : a \in A\}$ is a collection of topological spaces. Suppose moreover that for each $a \in A, f_a : X \to Y_a$ is a function from X into Y_a. The **weak**

topology by $\boldsymbol{F}$, where $\boldsymbol{F} = \{f_a : a \in A\}$, is the topology generated by the collection

$$\boldsymbol{C} = \cup\{f^{-1}(\boldsymbol{T}_a) : a \in A\},$$

where $f^{-1}(\boldsymbol{T}_a) = \{f^{-1}(U) : U \in \boldsymbol{T}_a\}$. The weak topology by $\boldsymbol{F}$ is usually denoted $w(X, \boldsymbol{F})$,

Theorem 2.7. If $(X, \boldsymbol{T})$ is a topological space, $\{(Y_a, \boldsymbol{T}_a) : a \in A\}$ is a collection of topological spaces, and $\boldsymbol{F} = \{f_a : a \in A\}$ is a collection of functions $f_a : X \to Y_a$, then each f_a is continuous if and only if $\boldsymbol{T}$ includes the weak topology by $\boldsymbol{F}$.

Proposition 2.8. Suppose X is a set and $\{(Y_a, \boldsymbol{T}_a : a \in A\}$ is a collection of topological spaces. Suppose further that for each $a \in A$, $f_a : X \to Y_a$ is a function from X into Y_a, and that for each $a \in A$, the collection $\boldsymbol{B}_a$ is a base for $\boldsymbol{T}_a$. Let $\boldsymbol{B} = \cup\{f^{-1}(\boldsymbol{B}_a) : a \in A\}$. Then the topology generated by $\boldsymbol{B}$ is the weak topology by the collection $\boldsymbol{F} = \{f_a : a \in A\}$.

Proposition 2.9. Suppose (Y, d) is a pseudometric space, X is a set, and $f : X \to Y$ is a function. Then ρ defined by $\rho(x,y) = d(f(x),f(y))$ is a pseudometric for X, and the topology generated by ρ is the weak topology by f. [Strictly speaking, we should say the weak topology by $\{f\}$.]

Theorem 2.10. Suppose $f : Z \to X$ is a function from one topological space into another, and suppose the topology for X is generated by a collection of sets $\boldsymbol{C}$. Then f is continuous if and only if $f^{-1}(C)$ is open whenever $C \in \boldsymbol{C}$.

Theorem 2.11. Suppose $\{(Y_n, d_n) : n \in Z_n\}$ is a countable collection of pseudometric spaces, X is a set, and for each n, $f_n : X \to Y_n$ is a function from X into Y_n. Then the weak topology by the collection $\boldsymbol{F} = \{f_n : n \in Z_n\}$ is generated by a pseudometric.