Chapter Three - Connected Spaces

Definition. A topological space X is **disconnected** if there exist nonempty sets A and B such that $X = A \cup B$ and $A \cap clB = (clA) \cap B = \emptyset$. A space that is not disconnected is said to be **connected**.

Theorem 3.1. A space is disconnected if and only if it has a proper subset that is both open and closed.

Proposition 3.2. The interval $[0,1]$ is connected. (Usual topology, of course.)

Proposition 3.3. Let $\Gamma = \{0,1\}$ be the two point discrete space. A space X is disconnected if and only if there is a continuous function from X onto Γ.

Theorem 3.4. Suppose $\mathcal{C}$ is a collection of connected subsets of a space and suppose there is a $C^* \in \mathcal{C}$ such that $C \cap C^* \neq \emptyset$ for every $C \in \mathcal{C}$. Then $\cup\mathcal{C}$ is connected.

Theorem 3.5. Suppose $\mathcal{C}$ is a collection of connected subsets of a space and suppose $C \cap D \neq \emptyset$ for all $C, D \in \mathcal{C}$. Then $\cup\mathcal{C}$ is connected.

Examples 3.6.
a)The real line with the usual topology is connected.
b)Euclidean n –space is connected.

Theorem 3.7. If A is a connected subset of a space and $A \subset B \subset clA$, then B is connected.

Theorem 3.8. If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.

Proposition 3.9. Let X be a topological space and define the relation R on X by

$$R = \{(a,b) : \text{there is a connected subset of } X \text{ containing } a \text{ and } b.\}$$

Then R is an equivalence relation.

Definition. For the equivalence relation R defined in Proposition 3.9, the equivalence classes are called **components** of X.

Theorem 3.10. Each component of a space is closed and connected.

Definition. A continuous function $f : [0,1] \to X$ is a **path** in X.

Definition. A space X is **path connected** if for each x and y in X, there is a path f in X such that $f(0) = x$ and $f(1) = y$.

Theorem 3.11. Every path connected space is connected.

Theorem 3.12. Every open connected subset of Euclidean n −space is path connected.

Theorem 3.13. If X is path connected and $f : X \to Y$ is continuous, then $f(X)$ is path connected.